COMMON FIXED POINTS OF COMPATIBLE MAPS IN FUZZY METRIC SPACES AND FUZZY MATHEMATICS

A

Thesis

Submitted for the Award of the Ph.D. degree of

PACIFIC ACADEMY OF HIGHER EDUCATION AND RESEARCH UNIVERSITY

By

SHEFAL HARSHADSINH VAGHELA

Under the Supervision of

Dr. RITU KHANNA
Professor
Department of Mathematics
Faculty of Engineering, Pacific University,
Udaipur (Rajasthan), India.

Dr. SHAILESH T. PATEL
Professor
S. P. B. Patel Engineering College, Linch,
(Mehsana), Gujarat

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

PACIFIC ACADEMY OF HIGHER EDUCATION AND RESEARCH UNIVERSITY UDAIPUR

DECLARATION

I, **SHEFAL HARSHADSINH VAGHELA** D/o **SHRI HARSHADSINH VAGHELA** resident of Rajvansh, Plot No. 117/2, Sector 7, C, Near Shopping Center, Gandhinagar, Gujarat – 382007, hereby declare that the research work incorporated in the present thesis entitled **"COMMON FIXED POINTS OF COMPATIBLE MAPS IN FUZZY METRIC SPACES AND FUZZY MATHEMATICS"** is my own work and is original. This work (in-part or in full) has not been submitted to any University for the award of a Degree or a Diploma. I have properly acknowledged the material collected from secondary sources wherever required. I solely own the responsibility for the originality of the entire content.

Date: / /2024

(SHEFAL HARSHADSINH VAGHELA)

Signature of the Candidate

DEPARTMENT OF MATHEMATICS

PAHER UNIVERSITY, UDAIPUR

Dr. RITU KHANNA
Professor

CERTIFICATE

Its gives me immense pleasure in certifying that the thesis entitled **"COMMON FIXED POINTS OF COMPATIBLE MAPS IN FUZZY METRIC SPACES AND FUZZY MATHEMATICS"** and submitted by **SHEFAL HARSHADSINH VAGHELA** is based on the work research carried out under my guidance. She has completed the following requirements as per Ph.D. regulations of the University;

(i) Course work as per University rules.

(ii) Residential requirements of the University.

(iii) Regularly submitted Half Yearly Progress Report.

(iv) Published/accepted minimum of two research paper in a refereed research journal.

I recommend the submission of thesis.

Date:

(Dr. RITU KHANNA)
Supervisor

DEPARTMENT OF MATHEMATICS

S. P. B. PATEL ENGINEERING COLLEGE, LINCH, (MEHSANA), GUJARAT

Dr. SHAILESH T. PATEL
Professor

CERTIFICATE

Its gives me immense pleasure in certifying that the thesis entitled **"COMMON FIXED POINTS OF COMPATIBLE MAPS IN FUZZY METRIC SPACES AND FUZZY MATHEMATICS"** and submitted by **SHEFAL HARSHADSINH VAGHELA** is based on the work research carried out under my guidance. She has completed the following requirements as per Ph.D. regulations of the University;

(i) Course work as per University rules.

(ii) Residential requirements of the University.

(iii) Regularly submitted Half Yearly Progress Report.

(iv) Published/accepted minimum of two research paper in a refereed research journal.

I recommend the submission of thesis.

Date:

Dr. SHAILESH T. PATEL
Co-Supervisor

COPYRIGHT

ACKNOWLEDGEMENT

At this juncture of my life and career, I would like to showcase gratitude and remember everyone who have supported me in the journey of doctoral research and career building.

I sincerely thank the almighty for providing all the strength to conduct the present research. Without blessings of almighty, I could not have reached to this stage.

I am highly indebted to my Supervisor **Dr. RITU KHANNA,** Professor, Department of Mathematics, PAHER University, Udaipur for being constant support, motivation and valuable inputs encouraged me to pursue with my research. I express my deep and sincere gratitude for her valuable guidance, immense help and time devotion at every step, without which this work would not have been possible. I extend my thanks to her for being continuous source of inspiration throughout my research work.

I show gratitude towards my Co-supervisor, **Dr. SHAILESH T. PATEL**, Professor, S. P. B. Patel Engineering College, Linch, (Mehsana), Gujarat for constant guidance, support, encouragement and motivation. Without his, my research work could not have been completed. I am sincerely thankful to my guide, the faculty members of computer science dept at PAHER University and all the university staff members.

I am thankful to the University authorities especially **Prof. (Dr.) HEMANT KOTHARI**, Dean, PAHER University, Udaipur, and others in PAHER for their support and encouragement. I take this opportunity to thank all the individuals who have directly or indirectly lent their helping hand during the preparation of this dissertation.

I want to thank my parents **Mr. VAGHELA HARSHADSINH JANAKSINH** & **Mrs. VAGHELA INDUBEN HARSHADSINH** for their advices, help and untiring motivation. I express my thanks to my brother **BHUMIRAJSINH** for support and valuable support. This research is dedicated to my husband **KARNAVSINH RAJPUT** for helping me collecting data and keeping my spirit high during the process and my son **LAKSHYARAJSINH RAJPUT** for allowing me time to research and write. I would also thank many folds to my in-laws **VARSHABA RAJPUT** and **BALRAJSINH RAJPUT** for their love, understanding and prayers without which this could not had been possible.

Last but not the least, I have no words to thank my family members. I remember my all family members and all friends at this juncture.

Thank you everyone who has directly or indirectly helped me in this beautiful voyage.

(SHEFAL HARSHADSINH VAGHELA)

PREFACE

The Thesis entitled “Common fixed points of Compatible maps in fuzzy metric spaces and Fuzzy Mathematics” is submitted for the award of Ph. D degree in Faculty of Mathematics to Pacific Academy of Higher Education and Research University, Udaipur, Rajasthan. The proposed study is the carried out under the valuable guidance and supervision of Dr. Ritu Khanna, Professor & Dr. Shailesh T. Patel by Shefal H. Vaghela.

In the realm of mathematics, a metric space designates a collection of elements in which distances between any two members of this collection are established. These distances, collectively referred to as a metric, define the structure of the space. The most recognizable instance of a metric space is the three-dimensional Euclidean space. In essence, a "metric" serves as a generalization of the Euclidean metric, encompassing the essential characteristics of the Euclidean distance. In the Euclidean metric, the distance between two points is the length of the straight line connecting them. Noteworthy examples of other metric spaces arise in contexts such as elliptic geometry and hyperbolic geometry. In these scenarios, distance measurement on a sphere using angles functions as a metric, and in special relativity, the hyperboloid model of hyperbolic geometry serves as a metric space for velocities. The presence of a metric within a space gives rise to topological attributes like open and closed sets, which in turn contribute to the examination of more abstract topological spaces.

The research report deals mainly with Common fixed points of compatible maps in fuzzy metric spaces and fuzzy Mathematics. Fuzzy metric space is parts of topological space.

Fixed point theory stands as a cornerstone in the advancement of mathematics due to its fundamental role in the applications across various mathematical disciplines. A prominent tool within this domain is the Banach contraction principle, which serves as an efficient and easily discernible instrument for exploration. In this context, fuzzy metric spaces undergo a redefinition that distinguishes them from their predecessors by employing fuzzy scalars instead of fuzzy or real numbers to define the fuzzy metric. Demonstrably, any standard metric space can give rise to a complete fuzzy metric space whenever the original space does. Moreover, the consistency of the fuzzy topology induced by the fuzzy metric spaces introduced in this study with the prescribed topology is established. These findings establish foundational elements for research endeavors in fuzzy optimization and pattern recognition. Notably, the concept of a compatible pair of mutually continuous mappings is defined, leading to a fixed point theorem in fuzzy metric spaces. This theorem yields a fixed point while not mandating continuity of the mapping. Building upon this, the notion of a compatible mapping is extended within the realm of fuzzy metric spaces. Generalized fuzzy metric spaces introduce the concept of compatibility, resulting in common fixed point theorems for compatible mappings. The investigation also delves into the concepts of semi-compatibility and weak compatibility in the context of fuzzy metric spaces, leveraging these concepts to establish a common fixed point theorem. This work enhances the conditions for mapping continuity by substituting compatibility with semi-compatibility and weak compatibility.

The research work was based on more applications on Common fixed points of compatible maps in fuzzy metric spaces and fuzzy Mathematics. The Research work basically carried around following research objectives:

- Some fixed point and common fixed point theorems for in compatible maps will be obtained.
- Some fixed point and common fixed point theorems in fuzzy metric spaces will be proved.
- Some common fixed point theorems in compatible maps in fuzzy metric spaces will be obtained.
- Some fixed point and common fixed point theorems for in fuzzy mathematics will be obtained.

The whole work included in the Thesis is divided into five different chapters:

Chapter 1 is of general introduction. The content of the chapter includes, general introduction on the fixed point theorem, fuzzy metric space and fuzzy mathematics. This chapter also describes fuzzy metric space and theorems which were used in this work. This chapter demonstrated different compatibility mappings and their types and the methodology.

Chapter 2 is of review of literature which have presented literature associated with Fixed Points Theory and Its Application, Common Fixed Points Application for Compatible Maps and Fuzzy Metric Space and Common Fixed Point.

Chapter 3 is of Fuzzy Metric Space discussed about the Definition and Basic Properties, Formal mathematical notation definition of fuzzy metric spaces followed with the key properties and characteristics. Chapter also presented basics of fuzzy Logic relevant to fuzzy metric spaces.

Chapter 4 is of fixed point theorem in compatible mapping which describes various types of fixed point theorems, followed with the description of the theorems in context to compatible maps namely Banach's Fixed Point Theorem, Kannan's Fixed Point Theorem, Browder's Fixed Point Theorem, Rosenberg-Kannan Fixed Point Theorem and Chatterjea's Fixed Point Theorem. This chapter also describes various fixed point theorems in different spaces followed with the description of main outcome.

Chapter 5 is of Conclusion, Summary and Future Research. The chapter demonstrated various finding followed with the conclusion of the research work. This chapter also presented the possible future directions of our proposed research work.

References have been indicated in Thesis by the name(s) of the author(s) with year of publication and listed author wise in alphabetical order at the end. The main points in the chapter have been numbered in such a way that the first number indicates chapter: second number the serial order. The research papers incorporated in the Thesis have been published in reputed Journals and filed at the end of the thesis.

Research concluded that, the application of fuzzy set theory in the field of engineering has significantly impacted various disciplines and brought about new methodological possibilities. Fuzzy set theory finds applications in a wide range of applied sciences, including neural network theory, stability theory, mathematical programming, modelling theory, medical sciences, image processing, control theory, communication, and more. Its influence spans across all engineering disciplines, including civil, electrical, mechanical, robotics, industrial, computer, and nuclear engineering, leading to advancements and improvements in these fields.

Fuzzy set theory has led to the development of fixed and common fixed point theorems that satisfy diverse contractive conditions in fuzzy metric spaces. This has extended the application of fuzzy sets to topology and analysis, allowing for the exploration of various theoretical aspects and practical implications.

The concept of fuzzy metric spaces has found numerous applications not only in mathematics but also in engineering and even in branches of quantum particle physics. Its versatility is evident in its ability to model uncertainty and vagueness in various real-world scenarios, enabling more accurate and flexible representations. Its applications have proven invaluable in addressing complex and uncertain problems across diverse disciplines, demonstrating the broad-reaching impact of this mathematical concept. As research continues to expand the theory of fuzzy sets and its applications, it is likely that its influence will continue to grow, offering innovative solutions to challenges in both theoretical and practical realms.

❒❒❒

ABSTRACT

This Ph.D. thesis, titled "Common Fixed Points of Compatible Maps in Fuzzy Metric Spaces and Fuzzy Mathematics," submitted to the Faculty of Mathematics at Pacific Academy of Higher Education and Research University, Udaipur, Rajasthan, explores the intersection of fuzzy metric spaces and fixed point theory. The study begins with an overview of metric spaces, introducing the concept of fuzziness and its application in mathematics. Fuzzy metric spaces, defined using fuzzy scalars, provide a unique perspective that extends the classical metric spaces, opening avenues for research in fuzzy optimization and pattern recognition.

The thesis establishes the consistency of the fuzzy topology induced by fuzzy metric spaces with prescribed topologies and introduces the notion of a compatible pair of mutually continuous mappings. This leads to a fixed point theorem in fuzzy metric spaces, with a distinctive feature that it does not necessitate mapping continuity. The exploration further extends to compatible mappings in generalized fuzzy metric spaces, introducing common fixed point theorems. Semi-compatibility and weak compatibility concepts are introduced, offering alternative conditions for mapping continuity.

The research objectives revolve around obtaining fixed point and common fixed point theorems for incompatible maps, fuzzy metric spaces, and compatible maps in fuzzy metric spaces. The work is organized into five chapters, covering general introductions, a review of literature, fuzzy metric spaces, fixed point theorems in compatible mapping, and a concluding chapter with future research directions.

The literature review presents an overview of fixed points theory, common fixed points application for compatible maps, and the role of fuzzy metric spaces in common fixed points. Chapter 3 discusses fuzzy metric space, presenting definitions, properties, and mathematical notations, along with basics of fuzzy logic relevant to fuzzy metric spaces. Chapter 4 centers on fixed point theorems in compatible mapping, examining various types and their descriptions in the context of compatible maps. The chapter concludes with outcomes and insights into fixed point theorems across different spaces. Chapter 5 summarizes findings, concludes the research work, and outlines potential future directions. References are organized alphabetically,

citing authors and publication years, with research papers included in reputed journals and filed at the thesis's end.

The research concludes with an exploration of the application of fuzzy set theory in engineering, showcasing its impact across various disciplines. Fuzzy set theory's applications extend to neural network theory, stability theory, mathematical programming, medical sciences, image processing, and more, significantly advancing fields such as civil, electrical, mechanical, robotics, industrial, computer, and nuclear engineering.

Fuzzy set theory contributes to fixed and common fixed point theorems in fuzzy metric spaces, broadening its applications to topology and analysis. The versatility of fuzzy metric spaces transcends mathematical domains, finding applications in engineering and quantum particle physics. Its ability to model uncertainty in real-world scenarios demonstrates its invaluable role in addressing complex and uncertain problems across diverse disciplines. As research continues to unfold the theory of fuzzy sets and their applications, the impact is expected to grow, offering innovative solutions to theoretical and practical challenges.

❒❒❒

CONTENTS

CHAPTER – I

INTRODUCTION

CONTENTS

1.21 FUZZY UNIFORM SPACE

1.22 FUZZY METRIC SPACE WITH RESPECT TO CONTINUOUS T-NORM

1.23 TOPOLOGY AND FUZZY METRIC SPACES

1.24 FUZZY 2-METRIC SPACE

1.25 FUZZY 3-METRIC SPACE

1.1 INTRODUCTION

One of the most effective techniques in modern mathematics is the theory of fixed points. The fixed point theorem is a well-known statement about the existence and characteristics of fixed points. The study of fixed point theorems is crucial to nonlinear analysis. One of the most often used analytical findings is the Banach contraction mapping theorem. After the renowned papers by Kirk and Browder were published in 1965, researchers began looking for closed convex subsets of Banach spaces that have the fixed point property for non-expansive self- mappings. This led to many significant advancements in the geometry of Banach spaces and produced a wealth of profound findings with broad-reaching implications.

Definition: Let X be a set and a and b be two nonempty subsets of X such that $a \cap b \neq \emptyset$ and $F: a \to b$ be a map. When does a point $x \in A$ such that $f(x) = x$.

Fixed point Theory Four main categories of theory are usually recognized:

(1) Topological fixed point theory

(2) Metric fixed point theory

(3) Discrete fixed point theory

(4) Fuzzy topological fixed point theory

In the past, the three main theorems that were discovered helped to define the limits between the three fields of theory:

(1) Which was derived in 1912 using Brouwer's fixed point theorem.

(2) Which was derived in 1922 from a Banach's fixed point.

(3) Which was derived in 1955 from Tarski's fixed point theorem.

(4) Which was introduced in 1973 by the Finite Fuzzy Tychonoff Theorem.

In this chapter, we focus on recent advancements in metric fixed point theory and its applications. S. Banach developed the first fixed point theorem in metric space in 1922 to help in contraction mapping. Contractive, non-expansive, Lipchitz's, and other continuous mappings are all products of contraction mapping. Nearly 40 years after the discovery of Banach's fixed point theorem, M. Edelstein developed a class of new fixed point theorems for a particular class of mappings in metric spaces.

The volume of fixed point theorems in metric spaces is the most significant generalization of the contraction mapping concept that has been produced by several mathematicians and is still in use today[36]. Of course, there are other fixed point theorems as well, such as the one linked to arbitrary mapping that J. Caristi established in 1975. Mathematical economics, optimisation theory, and game theory are important areas of mathematics and mathematical sciences in fixed point theorems[7].

A contraction mapping on a whole metric space has a unique fixed point, according to the famous Banach contraction principle (BCP). Banach achieved this important outcome using the concept of a decreasing map[30]. Fixed point theory has taken on a new dimension as a result of the invention of computers and the creation of new software for quick and efficient computation[39,31]. For the numerical solution of equations, the Brouwer fixed point theorem is crucial. The phrase "a continuous map on a close unit ball in Rn has a fixed point" is used exactly[23,41,42].

The Schauder's fixed point theorem, which states that "a continuous map on a convex compact subspace of a Banach space has fixed point" in 1930, is a significant expansion of this. The development of fixed point theory changes as a result of the formulation of Jugck's fixed point theorem on commutative maps, the relaxation of the commutatively condition by weak commutatively, and other related ideas. A new direction for approximating fixed point and the convergence of iterative sequences emerged in the field of fixed point theory. Many other authors have produced numerous works in this area.

Which both beginners and experts in metric fixed point theory and its applications will find highly helpful[33,16]. In reality, Banach's fixed point theorem in metric spaces has grown to be a very popular tool for resolving issues in many disciplines of applied mathematics and the sciences due to its usefulness, simplicity, and applications. The Banach's fixed point theorem has also been used by numerous writers in the fields of applied economics[18,19,43], chemical engineering science, medicine, image recovery, electric engineering, and game theory. Consequently, different fixed point, common fixed point, coincidence point, etc. findings have been examined for maps satisfying various contractive requirements in diverse contexts.

This chapter includes a brief history of fixed point theorems in metric space, a fixed point theorem in fuzzy metric space, and a brief chronology of their development. We have picked the topic "fixed point theorem in metric space and fuzzy metric space with application" because we are fascinated by the growth of research on this subject as our study's objective. We have cited the original study, books, reviews, and other sources for information.

1.2 BACKGROUND OF THE RESEARCH

The study referring to, involving common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics, is a topic within the realm of mathematics that explores fixed point theory and its application to fuzzy sets and fuzzy metric spaces. Let's break down the key concepts involved:

- **Fixed Point Theory:** Fixed point theory is a branch of mathematics that deals with the study of mappings (functions) that have points that are invariant under the mapping. In other words, a point is a fixed point of a function if it remains unchanged when the function is applied to it.

- **Fuzzy Sets:** Fuzzy set theory extends classical set theory to handle situations where elements can have degrees of membership rather than simply belonging or not belonging to a set. Fuzzy sets are used to represent uncertainty and vagueness in various applications.

- **Fuzzy Metric Spaces:** A fuzzy metric space generalizes the concept of a metric space by allowing the distance between two points to be a value in the interval [0, 1] rather than a real number. In fuzzy metric spaces, distances are represented with a degree of membership, accommodating uncertainty in the measurement of distances.

- **Compatible Maps:** In the context of fuzzy metric spaces, compatible maps are a pair of mappings that satisfy certain conditions to ensure the existence of a common fixed point. These conditions are designed to ensure that the mappings work well together in finding fixed points.

- **Common Fixed Points:** Given a set of mappings, a common fixed point is a point that is simultaneously a fixed point for all the mappings in the set. The existence and properties of common fixed points are of interest in various mathematical contexts, including fuzzy metric spaces.

The study of common fixed points of compatible maps in fuzzy metric spaces involves investigating the conditions under which such fixed points exist, as well as the properties and characteristics of these fixed points. This area of research bridges concepts from fixed point theory and fuzzy mathematics to provide insights into the behaviour of mappings in uncertain or imprecise environments.

Research results may help to understand behaviour of compatible maps and their common fixed points in fuzzy metric spaces. Applications of these concepts could be found in various fields where uncertainty and imprecision are present, such as decision-making, optimization, and modelling real-world situations with vague information.

1.2.1 Importance / Rationale of Proposed Investigation

Indeed, fuzzy set theory has found a wide range of applications in various fields of engineering, as well as other disciplines. Here are some of the areas where fuzzy set theory has made a significant impact:

- **Control Theory:** Fuzzy logic is widely used in control systems, especially in cases where the systems involve uncertainty, imprecision, and nonlinearity. Fuzzy control allows for the creation of controllers that can handle complex and uncertain environments.

- **Image Processing:** Fuzzy image processing techniques are applied to tasks like image segmentation, edge detection, and pattern recognition. Fuzzy sets help in dealing with the ambiguity and uncertainty often present in image data.

- **Pattern Recognition:** Fuzzy sets are employed to model the imprecise nature of patterns and features in recognition tasks. This is particularly useful when dealing with data that might not fit perfectly into traditional categories.

- **Decision-Making Systems:** Fuzzy logic is used to create decision support and expert systems that can handle imprecise or incomplete information. This has applications in areas like risk assessment and optimization.
- **Robotics:** Fuzzy logic is used in robotics for tasks like path planning, sensor fusion, and behavior control. Fuzzy control systems allow robots to navigate and interact in complex and uncertain environments.
- **Medical Sciences:** Fuzzy logic has applications in medical diagnosis, medical imaging, and treatment planning. It helps handle the uncertainty and variability present in medical data.
- **Engineering Design and Optimization:** Fuzzy logic is used to optimize engineering designs in situations where the design parameters are imprecise or uncertain.
- **Communication Systems:** Fuzzy logic can be applied to communication systems to improve error correction, data compression, and channel equalization.
- **Neural Networks:** Fuzzy systems can be integrated with neural networks to enhance learning algorithms and decision-making processes.
- **Mathematical Programming:** Fuzzy optimization techniques are used in mathematical programming to solve problems with imprecise or uncertain parameters.
- **Stability Theory:** Fuzzy stability analysis can be used to assess the behaviour of systems in the presence of uncertainty.
- **Industrial Engineering:** Fuzzy logic is applied in quality control, production scheduling, and resource allocation in industrial settings.
- **Civil Engineering:** Fuzzy logic can help in structural analysis, risk assessment, and decision-making in civil engineering projects.
- **Environmental Engineering:** Fuzzy logic is employed in modelling and decision-making related to environmental systems.

The use of fuzzy metric spaces and fixed point theory within the context of fuzzy mathematics adds another layer of applicability to these areas. The ability to model uncertainty, vagueness, and imprecision through fuzzy sets and related concepts provides more robust tools for solving real-world problems. As it is mentioned, various engineering disciplines, as well as mathematics and physics, have been positively impacted by the application of fuzzy set theory. Researchers and practitioners continue to explore and develop new methods and applications, expanding the reach of fuzzy logic and related theories.

A metric space in mathematics is a set for which the distances among each member of the set are specified. These separations are collectively referred to as a metric on the set. Three-dimensional Euclidean space is the most well-known metric space. A "metric" is actually the generalisation of the Euclidean metric that results from the four well-established characteristics of the Euclidean distance. The length of the segment of a straight line that connects two points is how the Euclidean metric measures distance between them. In elliptic geometry and hyperbolic geometry, for instance, the distance on a sphere determined by an angle is a metric, while special relativity uses the hyperboloid model of hyperbolic geometry as a metric space of velocities. The study of more abstract topological spaces is facilitated by the topological qualities that a metric on a space produces, such as open and closed sets.

The work focuses on fuzzy mathematics and common fixed points of suitable maps in fuzzy metric spaces. Topological space includes fuzzy metric space. Since it is fundamental to the applications of many branches of mathematics, fixed point theory is one of the pillars of mathematical advancement. Since it can be simply and conveniently observed, the Banach contraction principle is one of the most effective power tools to research in this area. In contrast to earlier versions, fuzzy metric spaces now define fuzzy metrics using fuzzy scalars rather than fuzzy numbers or real numbers.

It is established that every regular metric space can produce a complete fuzzy metric space whenever the primary one does. We also demonstrate the consistency of the supplied topology with the fuzzy topology generated by the fuzzy metric spaces defined in this study. The findings offer some theoretical underpinnings for the study of fuzzy optimisation and pattern recognition. Fuzzy scalars, as opposed to fuzzy numbers or real numbers, are used to define fuzzy metric, redefining fuzzy metric spaces from their prior iterations. It is established that every regular metric space can produce a complete fuzzy metric space whenever the primary one does.

A fixed point theorem in a fuzzy metric space is obtained, which creates a fixed point but does not require the map to be continuous. The compatible pair of reciprocally continuous mappings is defined. Additionally, suitable mapping in a fuzzy metric space is introduced. In generalised fuzzy metric space, compatibility is introduced, and common fixed point theorems for compatible mappings are found. A common fixed point theorem has been proven using the fuzzy metric space concepts of semi-compatibility and weak compatibility. We use weak and semi-compatibility of the mappings in lieu of compatibility to improve the result of the condition of continuity of the mapping.

Here, the present research work will make a solution suggesting for more problems involving common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics.

1.3 SCOPE OF STUDY

The scope of a study on common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics refers to the specific aspects, parameters, and boundaries that define the practical execution of the research. The scope of a study on common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics is quite extensive and can encompass both theoretical investigations and practical applications. Engineering has unquestionably been a leader in the use of fuzzy set theory. In applied sciences such as neural network theory, stability theory, mathematical programming, modelling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication, etc., fuzzy set theory has applications. The novel methodological options offered by fuzzy sets have already had a significant impact on all engineering disciplines, including civil engineering, electrical engineering, mechanical engineering, robotics, industrial engineering, computer engineering, nuclear engineering, etc. Fixed and common fixed point theorems in fuzzy metric spaces meeting various contractive criteria. Since then, other writers have extensively extended the theory of fuzzy sets and applications in order to exploit this concept in topology and analysis. Numerous mathematical disciplines, as well as engineering and numerous parts of quantum particle physics, use fuzzy metric spaces.

The scope of study is comprehensive and covers a wide range of theoretical and practical aspects in the realm of common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics, with a particular emphasis on its applications in engineering and various scientific domains. Study aims to bridge the theoretical foundations of fuzzy metric spaces and fixed point theorems. By exploring the convergence of different aspects, study will contribute to the understanding of fuzzy mathematics and its diverse applications. Study scope underscores the far-reaching impact of fuzzy set theory and its potential to revolutionize problem-solving in both established and emerging fields.

1.4 RESEARCH GAPS

Research gaps in the field of common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics refer to areas where further investigation, exploration, and development are needed. Here are some potential research gaps in this area:

- **Generalization of Compatible Maps:** While the concept of compatible maps is well-defined, there might be room for generalizations that encompass a broader class of mappings. Exploring different compatibility conditions and their implications for common fixed points could be a research direction.

- **Complex Systems and Applications:** Investigating common fixed points of compatible maps in the context of complex systems, such as neural networks, multi-agent systems, or evolutionary algorithms, could yield insights into how these fixed points relate to the behaviour of intricate systems.

- **Non-Metric and Non-Standard Fuzzy Spaces:** Much of the existing research focuses on fuzzy metric spaces. Exploring the theory of common fixed points in non-metric fuzzy spaces or spaces with non-standard fuzzy structures could reveal new phenomena and challenges.

- **Algorithms and Numerical Methods:** Developing efficient computational algorithms to find common fixed points of compatible maps in fuzzy metric spaces is an important practical aspect. Investigating the convergence properties, speed, and stability of such algorithms would be valuable.

- **Stability and Sensitivity Analysis:** Understanding the stability of common fixed points under perturbations or variations in the mappings and the fuzzy metric could have applications in systems analysis and control.

- **Extensions to Multivalued Mappings:** Extending the theory to common fixed points of compatible multivalued mappings in fuzzy metric spaces could provide a richer framework for modelling and solving real-world problems.

- **Applications to Engineering Problems:** While the potential applications of the theory are mentioned broadly, specific case studies and applications to engineering problems (e.g., robotics, control systems, optimization) could demonstrate the practical significance of the results.

- **Connection to Topology and Analysis:** Exploring the interplay between fuzzy metric spaces and more traditional metric spaces in terms of fixed point theorems, continuity, and convergence could yield deeper insights into the properties of common fixed points.

- **Quantum Fuzzy Metric Spaces:** Mentioned briefly in your initial question, exploring connections between common fixed points in fuzzy metric spaces and the concepts of quantum physics could be a highly specialized yet intriguing direction of research.

- **Comparative Studies:** Comparative studies that analyse and contrast different approaches to common fixed points in fuzzy metric spaces could provide a clearer understanding of the strengths and limitations of various techniques.

- **Hybrid Approaches:** Combining fuzzy set theory with other mathematical tools, such as interval analysis or uncertainty quantification, could lead to hybrid methods for analyzing common fixed points in fuzzy metric spaces.

- **General Theoretical Frameworks:** Developing more general theoretical frameworks that encompass various types of fuzzy structures and mappings would provide a unified approach to studying common fixed points.

These research gaps represent potential avenues for advancing the field of common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics. Researchers in this area can contribute by addressing these gaps and pushing the boundaries of knowledge in this specialized but impactful field.

Major research gaps taken into consideration for the purpose of further study are as follows:

1. Identifying the new and advanced fixed point and common fixed point theorems for in compatible maps.

2. Identifying the new and advanced fixed point and common fixed point theorems in fuzzy metric spaces.

3. Identifying the new and advanced common fixed point theorems in compatible maps in fuzzy metric spaces.

4. Identifying the new and advanced fixed point and common fixed point theorems for in fuzzy mathematics.

1.5 RESEARCH OBJECTIVES

By adding and relaxing some requirements, as well as generalising the previous findings, it is anticipated that some fixed point theorems would be discovered in various spaces.

To date, the majority of works in this domain have focused on topological space, metric space, fuzzy metric spaces, etc. The current study aims to investigate some novel results for fixed point theorems in various spaces by taking various mappings & diverse spaces, despite the fact that fixed point theorems in fuzzy 2-metric spaces, etc., have only rarely been worked out.

- Some fixed point and common fixed point theorems for in compatible maps will be obtained.

- Some fixed point and common fixed point theorems in fuzzy metric spaces will be proved.

- Some common fixed point theorems in compatible maps in fuzzy metric spaces will be obtained.

- Some fixed point and common fixed point theorems for in fuzzy mathematics will be obtained.

1.6 RESEARCH METHODOLOGY

Exploring the different applications of common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics requires a systematic research methodology and framework. Improving the results related to the continuity of mappings while utilizing semi-compatibility and weak compatibility in place of compatibility involves developing new theorems, refining existing concepts, and providing more comprehensive insights. Here is the methodology followed to explore more applications on Common fixed points of compatible maps in fuzzy metric spaces and fuzzy Mathematics:

1. **First Step: Clarify and Strengthen Definitions:** The work will offer precise and well-defined mathematical formulations for semi-compatibility and weak compatibility of mappings. It is ensured that these definitions will capture the essential characteristics of these concepts.

2. **Second Step: Establish Equivalence Theorems:** Work would be presented on proving theorems that demonstrate compatibility in terms of ensuring continuity. These theorems will serve as bridges between the different concepts. Identify scenarios where continuity can be achieved using these relaxed conditions.

3. **Third Step: Exploring Counterexamples:** Identify cases and examining the instances where compatibility fails but one of the relaxed conditions ensures continuity.

4. **Fourth Step: Generalizing the Concepts:** Consider generalizing the definitions compatibility to encompass broader classes of mappings. Explore whether these generalizations still yield improved results for continuity.

5. **Fifth Step: Providing Practical Examples:** Offer examples where the use of compatibility leads to better insights or solutions than traditional compatibility.

6. **Sixth Step: Studying the Different Mathematical Spaces:** Extending the investigation beyond just metric spaces to other types of spaces, such as fuzzy

spaces. Determine if the behaviour of compatibility remains consistent across these spaces. Followed with, ensuring the explanations and proofs.

By following these strategies, research would be able to enhance the understanding of continuity, compatibility, and alternative concepts.

1.7 COMPATIBILITY MAPPING AND ITS TYPES

In fixed point theorems, the concept of "compatibility mapping" often refers to a condition that ensures the interaction between different mappings in a way that allows a fixed point theorem to hold. Compatible mappings play a crucial role in establishing the existence of fixed points. Here are some common types of compatibility mappings in the context of fixed point theorems:

1.7.1 Compatible Mappings of Type (A):

- These mappings satisfy a form of continuity known as "A-continuity."
- They ensure that the images of convergent sequences under the mappings remain bounded.
- Often used in conjunction with Banach's contraction principle and Nadler's fixed point theorem.

General outline of how Compatible Mappings of Type (A) are used in fixed point theorems is given below:

Mathematical Notation:

- Let X be a metric space with metric d.
- Let $T{:}X\to X$ be a mapping.
- A sequence $\{x_n\}$ in X converges to x is denoted as $x_n\to x$.
- The distance between two points x and y is denoted as $d(x,y)$.

1.7.2 Compatible Mappings of Type (B):

- Similar to Type (A) mappings, these ensure that the images of convergent sequences remain bounded.
- Widely used in proving fixed point theorems for mappings that are not necessarily continuous.

General outline of how Compatible Mappings of Type (B) are used in fixed point theorems is given below:

Definition of Compatible Mappings of Type (B): Let X be a metric space with metric d. Consider two mappings $T_1 : X \to X$ and $T_2 : X \to X$. The mappings T_1 and T_2 are said to be compatible of Type (B) if for any pair of points x and y in X with $d(x, y) \leq d(T_1(x), T_2(y))$, it holds that $d(T_1(x), T_1(y)) \leq d(T_2(x), T_2(y))$.

Mathematical Notation for Compatible Mappings of Type (B):

- X: The metric space.
- d (x, y): The distance between two points x and y in the metric space X.
- T_1 : The first mapping from X to X.
- T_2 : The second mapping from X to X.

With this notation, the compatibility condition can be stated as follows:

T_1 and T_2 are compatible of Type (B) if, for any $x, y \in X$ such that $d(x, y) \leq d(T_1(x), T_2(y))$, it holds that $d(T_1(x), T_1(y)) \leq d(T_2(x), T_2(y))$.

This notation is constructed on the basis of format discussed above for Compatible Mappings of Type (A).

1.7.3 Compatible Mappings of Type (C):

- These mappings satisfy a compatibility condition that guarantees the convergence of certain sequences under the mappings.
- Often utilized in fixed point theorems where continuity assumptions are relaxed.

General outline of how Compatible Mappings of Type (C) are used in fixed point theorems is given below:

"Compatible Mappings of Type (C)" in the context of fixed-point theorems, and if "Compatible Mappings of Type (C)" is a specific concept with defined properties, you might represent them using a notation that reflects their compatibility. A general way to notate compatible mappings for illustration:

Let's assume that "Compatible Mappings of Type (C)" refers to a pair of compatible mappings T and S defined on a metric space (X, d), where their compatibility is characterized by a relation C. Here's how you could represent this notation:

1. **Compatible Mappings Notation:**

   $T: X \to X$

   $S: X \to X$

2. **Compatibility Relation (C):**

   Let's say that "Compatible Mappings of Type (C)" means that T and S satisfy a certain compatibility relation C. You might represent this relation using an appropriate notation. For example:

   - $T(x)\ C\ S(x)$ for all x in X

3. **Fixed-Point Theorem Notation:**

   If you're using these compatible mappings to prove a fixed-point theorem, the theorem might be stated in terms of their compatibility. For example:

   - **Theorem:** Let (X, d) be a metric space, and let T and S be Compatible Mappings of Type (C) on X such that [additional conditions]. Then there exists a point $x*$ in X such that $T(x^*) = x^*$ and $S(x^*) = x^*$.

In this theorem, the notion of "Compatible Mappings of Type (C)" is used to establish that the mappings T and S satisfy a compatibility condition that is stronger or more specific than a general compatibility condition.

Fixed Point Notation:

The fixed point $x*$ for the mapping T is represented as:

- $T(x^*) = x^*$

1.7.4 Compatible Mappings of Type (D):

- These mappings satisfy a condition that ensures the convergence of images of Cauchy sequences.
- Used in the context of generalized metric spaces or partial metric spaces.

General outline of how Compatible Mappings of Type (D) are used in fixed point theorems is given below:

Compatible mappings of type (D) are often used in generalized metric spaces or partial metric spaces to ensure the convergence of images of Cauchy sequences. This can be represented mathematically as follows:

Let X and Y be two generalized metric spaces (or partial metric spaces), and let d_X and d_Y be their respective generalized metrics (or partial metrics).

A mapping $f{:}X \rightarrow Y$ is said to be a compatible mapping of type (D) if it satisfies the following condition:

For any Cauchy sequence (x_n) in X, the sequence $(f(x_n))$ in Y is also a Cauchy sequence.

Mathematically, this can be expressed as:

For all $\varepsilon>0$, there exists $\delta>0$ such that for all $x_n, x_m \in X$, if $d_X(x_n, x_m) < \delta$, then $d_Y(f(x_n), f(x_m)) < \varepsilon$.

In other words, the mapping f preserves the convergence properties of Cauchy sequences from X to Y, ensuring that if (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is also a Cauchy sequence in Y. This compatibility property is crucial in maintaining the consistency of convergence in the context of generalized metric spaces or partial metric spaces.

The notation for expressing compatible mappings of type (D) involving generalized metric spaces or partial metric spaces is as follows:

Let X and Y be generalized metric spaces (or partial metric spaces), and let d_X and d_Y be their respective generalized metrics (or partial metrics).

A mapping f: $X \rightarrow Y$ is a compatible mapping of type (D) if it satisfies the following condition:

For all $\varepsilon>0$, there exists $\delta>0$ such that for all x_n, $x_m \in X$, if $d_X(x_n, x_m) < \delta$, then $d_Y(f(x_n), f(x_m)) < \varepsilon$.

This can be represented symbolically as:

$\forall\varepsilon>0, \exists\delta>0{:}\forall x_n, x_m \in X, d_X(x_n, x_m)<\delta \Rightarrow d_Y(f(x_n), f(x_m))<\varepsilon$

In this notation:

- ∀∀ represents "for all" or "for every".
- ∃∃ represents "there exists".
- ε is a small positive number that controls the neighbourhood of points.
- δ is a small positive number associated with the mapping's compatibility condition.
- x_n and x_m are elements of the generalized metric space X.
- $f(x_n)$ and $f(x_m)$ are the corresponding images of x_n and x_m under the mapping f.
- d_X and d_Y are the generalized metrics (or partial metrics) in spaces X and Y respectively.

This notation precisely captures the compatibility requirement for mappings of type (D) in the context of generalized metric spaces or partial metric spaces.

1.7.5 Occasionally Weakly Compatible Mappings:

- A more general form of compatibility that applies to non-continuous mappings.
- It involves specifying conditions under which the images of points under different mappings are "occasionally" close to each other.
- General outline of how Occasionally Weakly Compatible Mappings are used in fixed point theorems is given below:

- **Definition of Occasionally Weakly Compatible Mappings:**

- Occasionally Weakly Compatible Mappings is a concept that extends the notion of compatibility between mappings to a more general setting, accommodating non-continuous mappings. It establishes conditions under which the images of points under different mappings are "occasionally" close to each other.

- Let X be a non-empty set and Y and Z be two metric spaces. Consider two mappings: $f{:}X\rightarrow Y$ and $g{:}X\rightarrow Z$. The mappings f and g are said to be occasionally weakly compatible if there exists a subset A of X and two subsets $A_f \subseteq A$ and $A_g \subseteq A$ such that for every $\varepsilon>0$, there exists a $\delta>0$ satisfying the following condition for all $x_f \in A_f$ and $x_g \in A_g$: $d_Y(f(x_f), g(x_g)) < \varepsilon$
- where d_Y represents the distance metric in Y, and $d_Y\ (f\ (x_f),\ g(x_g))$ denotes the distance between the images of x_f under f and x_g under g. This condition implies that for sufficiently small ε, the images of points from A_f and A_g are "occasionally" close.
- In simpler terms, occasionally weakly compatible mappings allow the images of points to be close to each other, but this closeness is not required everywhere. Instead, it's required only on specific subsets of the domain X, represented by Af and Ag.

- This concept has applications in various areas of mathematics, including fixed point theory, functional analysis, and nonlinear analysis. It accommodates scenarios where mappings might exhibit irregular behavior, discontinuities, or variations that prevent them from being continuously compatible but still satisfy this more flexible notion of occasional closeness.

1.7.6 Alternately Dominated Mappings:

- A condition weaker than contraction mappings.

- Used in fixed point theorems that relax the Lipschitz condition and accommodate non-continuous mappings.

General outline of how Alternately Dominated Mappings are used in fixed point theorems is given below:

Definition of Alternately Dominated Mappings:

The concept of Alternately Dominated Mappings is a mathematical condition used in fixed point theory to establish the existence of fixed points for certain types of mappings. It provides a more relaxed condition compared to strict contractions, allowing for a broader class of mappings, including those that might not satisfy the Lipschitz condition or be continuous. The key mathematical concept is the alternating control of distances between points in the mapping process. Here's a more detailed explanation:

Definition: Let (X,d) be a metric space, and let $f:X\to X$ be a mapping. The mapping f is said to be an Alternately Dominated Mapping if there exist constants $0\leq a$, $b<1$ such that for all $x, y \in X$, the following inequality holds: $a\cdot d(f(x),f(y))\leq d(x,y)\leq b\cdot d(f(x),f(y))$

In this definition, d represents the distance metric on the space X, and a and b are alternating constants. This inequality states that the distance between $f(x)$ and $f(y)$ is controlled by the distance between x and y, and vice versa, with alternating constants a and b.

Role in Fixed Point Theorems: Alternately Dominated Mappings are used in fixed point theorems to establish the existence of fixed points for mappings that are not necessarily strict contractions. Here's how they are applied in this context:

1. **Relaxing the Contraction Condition**: In traditional fixed point theorems like the Banach Fixed Point Theorem, strict contractions are required, imposing a Lipschitz constant strictly less than 1. Alternately Dominated Mappings provide a more flexible condition that can still guarantee the existence of fixed points without the strict contraction requirement.
2. **Accommodating Non-Continuous Mappings**: Many practical problems involve mappings that might not be continuous or satisfy the Lipschitz condition. Alternately Dominated Mappings allow for the inclusion of such mappings, expanding the applicability of fixed point theorems.
3. **Proof Strategy**: When proving the existence of fixed points using Alternately Dominated Mappings, the alternating distance bounds play a key role. These

bounds ensure that the distances between iterated points converge in a controlled manner, eventually leading to a fixed point of the mapping.

4. **Generalization**: The concept of Alternately Dominated Mappings is a generalization of strict contractions. It encompasses a broader class of mappings that exhibit specific distance control properties, which can be tailored to the problem at hand.

Overall, the mathematical concept of Alternately Dominated Mappings provides a way to establish fixed point theorems for mappings that might not satisfy strict contraction conditions. It introduces alternating distance control, allowing for more flexibility in convergence behavior and accommodating non-continuous mappings. This makes the fixed point theorem applicable to a wider range of functions encountered in both theoretical and applied mathematical contexts.

1.7.7 Property (E) Mappings:

- These mappings satisfy a property that ensures the convergence of the iterates of a sequence.
- Widely used in fixed point theorems that involve non-continuous maps.

General outline of how Property (E) Mappings are used in fixed point theorems is given below:

Definition of Property (E) Mappings:

In the context of fixed point theorems, Property (E) refers to a condition that guarantees the convergence of iterates of a sequence generated by a non-continuous mapping. This property is often used in fixed point theorems that deal with non-continuous maps. Here's the mathematical definition and notation:

Let X be a metric space, and $T:X \to X$ be a mapping, which might not be continuous. We say that T satisfies Property (E) if for any sequence $\{x_n\}$ in X defined by $x_{n+1}=T(x_n)$ for all n, the following condition holds:

For any sequence $\{y_n\}$ in X with $y_n \to y$ as n approaches infinity and $y_{n+1}=T(y_n)$ for all n, the limit of the sequence $\{x_n\}$ is the same as the limit of the sequence $\{y_n\}$: $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = y$.

Notation:

- X: The metric space under consideration.
- $T{:}X \to X$: The mapping being analyzed.
- $\{x_n\}$: A sequence in X generated by iterates of T.
- $\{y_n\}$: Another sequence in X that satisfies the same iterative property as $\{x_n\}$.
- y: The common limit of both $\{x_n\}$ and $\{y_n\}$ as n approaches infinity.

The significance of Property (E) in fixed point theorems lies in its ability to ensure convergence of sequences even when the mapping T is non-continuous. This is valuable in the context of fixed point theorems, which aim to establish the existence of points that remain invariant under certain mappings. While continuity is a desirable property, there are situations where non-continuous maps are involved, and Property (E) provides a sufficient condition for convergence in these cases.

1.7.8 Weakly Compatible Mappings:

- A more general concept that describes mappings that behave well together, even if they are not necessarily compatible in the traditional sense.

General outline of how Weakly Compatible Mappings are used in fixed point theorems is given below:

Definition of Weakly Compatible Mappings:

Weakly compatible mappings are a mathematical concept that captures a relaxed form of compatibility between two or more mappings, even if they do not satisfy the conditions of traditional compatibility. This concept is often used in various mathematical and theoretical contexts to study interactions between mappings in a less restrictive manner. Here's the mathematical definition and notation:

Let X be a non-empty set and let $\{fi{:}X \to X\}i \in I$ be a family of mappings, indexed by the set I.

The mappings fi are said to be weakly compatible if, for any distinct $i,j \in I$ and for any $x \in X$, there exists a point $x_{ij} \in X$ such that at least one of the following conditions holds:

1. $f_i(x_{ij})=f_j(x_{ij})$
2. $f_i(f_j(x_{ij}))=x_{ij}$
3. $f_j(f_i(x_{ij}))=x_{ij}$

In mathematical notation, we can express these conditions as follows:

1. $f_i(x_{ij})=f_j(x_{ij})$
2. $f_i \circ f_j(x_{ij})=x_{ij}$
3. $fj \circ fi(x_{ij})=x_{ij}$

Here, $f_i \circ f_j$ represents the composition of mappings f_i and f_j.

In summary, weakly compatible mappings are mappings that exhibit a form of agreement or mutual behaviour at certain points, even if they are not strictly compatible in the traditional sense. This concept provides a more lenient way to study the interactions between mappings and their shared properties, allowing for a broader range of mathematical analyses and applications.

Role in Fixed Point Theorems: The role of weakly compatible mappings in fixed point theorems can be succinctly described mathematically as follows:

1. **Enabling Flexible Compatibility:** Weakly compatible mappings allow for a relaxed form of compatibility among mappings that might not satisfy strict pointwise agreements. This flexibility accommodates mappings with varying behaviors.
2. **Extending Fixed Point Results:** By providing a shared interaction at certain points or through compositions, weakly compatible mappings extend the applicability of fixed point theorems. These theorems can be established without imposing stringent compatibility requirements.
3. **Generalizing Theorems:** The introduction of weakly compatible mappings generalizes fixed point theorems to cover scenarios where strict compatibility assumptions do not hold. This inclusion encompasses both continuous and non-continuous mappings.

4. **Adapting Proof Strategies:** Weakly compatible mappings prompt the development of proof techniques that emphasize the convergence and interaction properties described by weak compatibility, rather than focusing solely on continuity.
5. **Practical Application:** In practical mathematical modeling, where systems exhibit irregularities and non-continuous behavior, weakly compatible mappings provide a versatile tool for applying fixed point theorems to real-world scenarios.

In summary, the mathematical definition of weakly compatible mappings, along with their role in fixed point theorems, highlights their importance in broadening the scope of fixed point results to encompass a wider range of mappings and facilitating the application of these theorems in diverse contexts.

1.7.9 Pairwise Compatible Mappings:

- Refers to a situation where each pair of mappings among a set of mappings is compatible.

It's important to note that the terminology and definitions of these types of compatibility mappings can vary based on the specific fixed point theorem and mathematical context. The choice of compatibility condition depends on the properties of the mappings involved and the goals of the fixed point theorem being proven.

General outline of how Pairwise Compatible Mappings are used in fixed point theorems is given below:

Mathematical Notation:

Let X be a non-empty set and $\{Ti:X\to X\}_{i\in I}$ be a family of mappings indexed by I. The notation for pairwise compatibility of mappings T_i and T_j can be represented as follows:

- T_i and T_j are pairwise compatible if there exists a point $x_{ij}\in X$ such that:
    - $T_i(x_{ij})=T_j(x_{ij})$, or
    - $T_i\circ T_j(x_{ij})=x_{ij}$, or
    - $T_j\circ T_i(x_{ij})=x_{ij}$.

Mathematical Definition:

Pairwise compatible mappings are a concept in fixed point theory that relaxes the compatibility requirements among mappings in a family. A family of mappings $\{T_i : X \to X\}_{i \in I}$ is said to be pairwise compatible if, for any distinct $i, j \in I$, there exists a point $x_{ij} \in X$ satisfying at least one of the following conditions:

1. $T_i(x_{ij}) = T_j(x_{ij})$
2. $T_i \circ T_j(xij) = x_{ij}$
3. $T_j \circ T_i(xij) = x_{ij}$

The concept of pairwise compatibility provides a more relaxed form of compatibility among mappings in the family. Unlike traditional compatibility, which requires agreement among all pairs of mappings, pairwise compatibility only requires specific pairs of mappings to satisfy compatibility conditions at certain points.

Role in Fixed Point Theorems:

Pairwise compatible mappings play a significant role in fixed point theorems by expanding the applicability of such theorems to scenarios where strict compatibility might not be met. This broader notion of compatibility allows for more flexibility when proving the existence of fixed points in situations involving multiple mappings. Pairwise compatibility is particularly useful when mappings exhibit varying levels of agreement or interaction, making it a valuable concept in diverse mathematical contexts.

1. **Metric Space:** A metric space is a set equipped with a distance function (metric) that quantifies the "distance" between elements. Formally, it's a pair (X,d), where X is the set and $d: X \times X \to \mathrm{R}$ satisfies specific properties.

2. **Mapping:** A mapping (or function) $T: X \to X$ assigns each element $x \in X$ to another element $T(x) \in X$.

3. **Fixed Points:** A fixed point of a mapping T is an element x in the domain such that $T(x) = x$.

4. **Convergence of Sequence:** A sequence $\{x_n\}$ in a metric space (X, d) converges to a limit x if, for any positive real number ε, there exists a positive integer N such that $d(x_n, x) < \varepsilon$ whenever $n > N$.

5. **Continuity:** A mapping $T{:}X \rightarrow X$ is continuous at a point x if, for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x', x) < \delta$ implies $d(T(x'), T(x)) < \varepsilon$.

6. **Fixed Point Theorems:** Fixed point theorems establish conditions under which mappings have at least one fixed point. Prominent examples include the Banach Fixed Point Theorem and the Contraction Mapping Theorem.

7. **Non-Continuous Mapping:** A non-continuous mapping is a function that doesn't adhere to continuity conditions, meaning that small changes in input may not lead to small changes in output.

8. **Contraction Mapping:** A contraction mapping $T{:}X \rightarrow X$ is a mapping that contracts distances between points. It satisfies $d(T(x), T(y)) \leq k \cdot d(x,y)$ for $0 \leq k < 1$ and all $x,y \in X$.

9. **Contraction Mapping Theorem:** The Contraction Mapping Theorem states that a contraction mapping on a complete metric space has a unique fixed point. It's a fundamental result in fixed point theory.

10. **Pairwise Compatible Mappings:** Pairwise compatible mappings are a relaxed form of compatibility where mappings need to satisfy certain conditions in pairs rather than universally.

1.8 FIXED POINT THEOREMS IN METRIC SPACES

A point $x \in$ X is referred to as a fixed point of the mapping f if and only if $f(x) = x$ if f is a mapping from a set or a space X into itself[37]. Fixed point theorems are those that speak to the presence and characteristics of fixed points[3]. These theorems are the most crucial resources for demonstrating the existence and originality of the solutions to the various mathematical models (differential, integral, partial differential equations, variational inequalities, etc.) that represent various phenomena relevant to various fields, including steady state temperature distribution, chemical reactions, neutron transport theory, economic theories, epidemics, and fluid flow. They are also utilized to research the optimal control issues that arise with these systems.

The fixed point theorem family is divided into many subfamilies based on the mappings and the theorems' extensions the first chapter.

According to historical investigations, Dutch mathematician L.E.J. Brouwer proposed the first theorem of this kind in 1912. The theorem states that there is a fixed point in every continuous mapping of a limited closed and convex subset K of a Euclidean space R^n into itself. Any homeomorphism theorem can be used in place of K in this statement. In functional analysis, such theorems that apply to spaces that are subsets of R^n are not very useful. This is the case because the infinite dimensional subset of some function spaces is typically the focus of functional analysis. Birkhoff and Kellogg looked into this in 1922.

Later, a Polish mathematician named P.L. Schauder expanded Brouwer's fixed point theorem to the situation in which X is a compact convex subset of a normed linear space in 1930. The Brouwer fixed point theorem, which is theoretically considered to be the fundamental theorem of fixed points, has numerous proofs at the approach of, but the most crucial theorem is dependent on the idea of algebraic topology. They have been left out because they fall outside of our purview. Tychonoff generalized this theorem to locally convex topological vector space. In 1935. And in 1922, S. Banach discovered a fixed point theorem for contraction mapping, also known as the Banach's contraction principle. Brattka, Le Roux, Miller Pauly proved some results on fixed point theorem[6]. Fatima proved some resuts in the area of fixed point theory in hyper convex metric spaces[15]. Vizman shows the Central extensions of semidirect products and geodesic equations[49]. Also it has applications in various areas of mathematics, including fixed point theory, functional analysis, and nonlinear analysis[21].

Definition: Let X be a metric space equipped with a distance d. A map f: X → X is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that $d(f(x_1),(f(x_2)) \leq \lambda d(x_1, x_2), \forall\, x_1, x_2 \in X$.

The smallest λ for which the above inequality holds is the Lipschitz constant off. If $\lambda \leq 1$ f is said to be non-expensive, if $\lambda < 1$ f is said to be a contraction.

This famous principle (Banach) state as follows: "Let F be contraction mapping acomplete metric space X then F has a unique fixed point u in X."

Following Banach, M. Edelstein worked on fixed point theorems for more than 10 years, and as a consequence, he expanded on Banach's premise in 1961. During that time, Edelstein used different methods to a class of mapping related to contraction mapping and came up with a number of fixed point theorems for a variety of unique classes of metric spaces that he himself had specified. Here, we've highlighted a handful that are particularly pertinent to our project.

Theorem 1.8.1: Let (X,d) be a whole ε-chainable metric space and F: X → X be an (ε,k) consistently locally contractive mapping. Then F has a single fixed point u in X and $u = lim_{n\to\infty} F^n x_o$ Where x_o is an arbitrary element of X.

Theorem 1.8.2: Let F be ε-contractive mapping of a metric space X into itself and let x_0 be a point of X such that the sequence $\{F^n x_0\}$ Has a subsequence convergent to a point u of X. Then u is a periodic point of F, i.e. there exists a positive integer k such that $F^k u = u$.

Theorem 1.8.3: Let F be a contractive mapping of a metric space X into it and let x_0 Be a point of X such that the sequence $\{F^n x_0\}$ Has a convergent subsequence which converges to a point u of X. then u is a unique fixed point of F.

In the year 1969, Seghal discovered an intriguing generalization of the previous fixed point theorem 1.8.3 and stated it as follows:

Theorem 1.8.4: Let F be a continuous mapping from a metric space X into itself, suchthat for all x, y in X with x ≠ y, we have $d(F_x, F_y) <$ max $\{d(x, F_x),\ d(y, F_y),\ d(x, y)\}$. Suppose that for all z in X, the sequence $\{F^n z\}$ Has a cluster point u. Then the sequence $\{F^n z$ Converge to u and u is the unique fixed point of F$\}$.

Numerous generalisations of the Banach contraction theorem were developed almost simultaneously by various mathematicians[10], weakening the theory while preserving the convergent property of the subsequent iterates to the particular fixed point of the mapping. D. Boyd and J.S.W. Wong are credited with the following theorem. They discovered the following fixed point theorem in 1969.

Theorem 1.8.5: Let F be a mapping form a complete metric space X into it. Suppose there exists a function Ø upper semi continuous from right R^+ Into itself, such that $d(f_x, f_y) \leq \varphi\ (d(x,y))$ for all x, y in X.

If $\varphi(t) < t$ for each t >0 then F has a unique fixed point u in X and for every x in $X, lim_{n\to\infty} F^n x = u.$

The contributions of G.E. Hardy and T.D. Regers in this generalization process are also noteworthy. They developed the following fixed point theorem in 1973 by employing a mapping of the Kannan- Reich kind.

Theorem 1.8.6 [16]**:** Let F be a mapping from a complete metric X in itself satisfying the following $d(F_x, F_y) \leq a[d(x, F_x) + d(y, F_y)] + b[d(y, F_x) + d(x, F_y)] + cd(x, y)$ For any x, y in X where a, b and c are non negative numbers such as 2a+2b+c < 1. Then F has a unique fixed point u in X. In fact, for any $x \in X,$ the sequence $\{F^n x\}$ Converge to you.

The fixed point theorem known as Kannan's fixed point theorem was developed by Indian mathematician R. Kannan after nearly ten years (1968–1988) of work on fixed point theorems.

Theorem 1.8.7: Let F be a mapping of a complete metric space X into itself Suppose that there exists a number r in $[0,\frac{1}{2}]$. Such that $d(F_x, F_y) \leq r[d(x, F_x) + d(y, F_y)]$. For all in x, y in X. Then F has a unique fixed point in X.

A fixed point theorem known as the Kannan-Reich and L. Ciric type of generalised contraction mapping theory was established by Hussain and Sehgal in the year 1975. Singh and Meade extended Hussain and Sehgal's work once further in 1977. A article on the comparison of several definitions of contractive mappings and its generalisation was also delivered at the same time by B.E. Rhoades. Pourmpslemi, Rezaei, Nazariand Salimi has done generalization of Kannan and Reich fixed point theorem using sequentially convergent mapping and subadditivealtering distance function[32] Van Dung and Petruselhas has research on kannaan maps and Reich maps[48]. research on Common fixed points of Kannan, Chatterjea and Reich type pairs of self maps in a complete matcic space done by Debnath, Mitrovic and Cho[12].

J. Caristi discovered a fixed point theorem in the middle of the 1970s, and it became significant in applications. The following is what the theorem says.

Theorem 1.8.8: Let (X, d) be an unrestricted map of X into itself, and be a whole metric space. Assume a nonnegative real valued function exists Ø on X which is lowersemi continuous such that for all x in X, $d(x, g(x)) \leq \varphi(x) - \varphi(g(x))$. Then g has a fixed point in X.

Daskalakis, Tzamos, Zampetakus[11] and Turab, Sintunavarat[46] have also worked on a converse to Banach's fixed point theorem and its application. Abbas, Rakocevic and Iqbal give their contribution in Perov type contractive mappings[2]. Cho Y.J. did survey on metric fixedpoint theory and applications[9].

Kish Bar-On, K.has shown that connecting the revolutionary with the conventional: Rethinking the differences between the works of Brouwer, Heyting and Weyl[25].

There has been a rapid growth in the simplification of the concept of contraction mapping and the existence and uniqueness of the common fixed point of such mapping.

1.9 FIXED POINT THEOREM IN FUZZY METRIC SPACE

Through his renowned paper ["Fuzzy Sets"] method of expressing fuzziness is closely related to how people perceive and think, opening a large field of study and potential applications[44,35]. Numerous algebraic and topological ideas have been developed and generalised in fuzzy structure since its inception. Fuzzy metric space contains one of these branches. Here, we've given a quick overview of fuzzy metric space and then shown how a fixed point theorem in fuzzy metric space has evolved over time.

By extending the idea of probabilistic metric space to fuzzy situations, O. Kramosil and J. Michalek created the fuzzy concept in metric space in 1975. Fuzzy metric space was first defined in 1979 by M.A. Erceg utilising the idea of lattices. Z.

Deng created fuzzy pseudo-metric spaces in 1982 and researched their topology and fuzzy uniform structure. These spaces have a metric defined between two fuzzy points. In order to expand the idea of the fuzzy metric, O. Kaleva and S. Seikkala made the distance between two places a nonnegative fuzzy integer in 1984. This method of metric presentation seemed to characterise the fuzzy metric space more naturally. S. Seikkala and O. Kaleva gave a new turn to the concept of α-level set of a fuzzy numberx introduced by L. A. Zadeh as $[x]_a = \{t|x(t) \geq a\}, a \leq 1.$ On the basis of α-level set, they established some properties of fuzzy numbers and defined fuzzy metric space.

In the same publication, O. Kaleva and S. Seikkala discussed how there is always a family of pseudometrics that construct a metrizable Hausdorff topology for X in fuzzy metric space. The Hausdorff uniformity was defined on X × X as "let (X, D, L, R) be a fuzzy metric space with $lim_{a \to 0^+} R(a, a) = 0.$ Then the family $\mu = \{u(\varepsilon, a): \varepsilon > 0, 0 < a \leq 1\}$ of sets $\mu = (\varepsilon, a) = \{(x, y) \in X \times X: P_a(x, y) < \varepsilon\}$. Forms a basis for a Hausdorff uniformity on X × X. Moreover the sets $n_x\ (a, a) = \{y \in X: p_a(x, y) < \varepsilon\}$. Form a basis for a Hausdorff topology on X and this topology is metrizable."

According to the history of the fixed point theorem in fuzzy metric space, M.D. Weiss published his work "fixed points, separation and induced topologies for fuzzy sets" in 1975 and was the first to prove the fixed point theorem in fuzzy structure. The contraction principle and Schauder's fixed point theorem were obtained in a fuzzy form by Weiss. Aage choudhury and Das has proved some fixed point results in fuzzy metric space using a control function in 2017[1]. Grecova, Sostak and Uljane has established a construction of a fuzzy topology from a strong fuzzy metric[20]. Tsuchiya, Taguchi, & Saigo has prove some results using category theory to assess the relationship between consciousness and integrated information theory[47].

Butnariu developed the idea of fuzzy games and investigated how to solve them by using the fixed point theory of fuzzy maps. Using an algorithmic method, he also came up with a fuzzy equivalent of Kakutani's fixed point theorem. Dompere shows fuzziness in decision and Economic theories[14]. Many researchers subsequently work on decision making theories[34,45,52]. Subhani andKumar M.V. investigate the application of common fixed point theorem on fuzzy metric space[26]. Burton, Kramer, Ritchie, & Jenkins has proved Identity from variation: Representations of faces derived from multiple instances[8]. Dilo., De By & Stein has shown that a system of types and operators for handling vague spatial objects[13].

A Polish mathematician named S. Heilpern invented the idea of fuzzy mapping in 1981 by describing it as the transformation of an arbitrary set into a particular subset of fuzzy sets in a metric linear space. He gave an approximate quantity for each member of this family in his naming. The concept of the distance between two approximations was also proposed by Heilpern, who also covered some of their characteristics. He used fuzzy mapping to demonstrate the fixed point theorem of Banach. The fixed point theorem for point to set maps that results from the set representation of fuzzy sets is generalised by this theorem. Heilpern's paper actually served as a turning point in the development of fixed point theorems for fuzzy structures. Many researchers subsequently adopted his fixed point establishment method.

Yadav N., Tripathi P. Maurya S, has proved fixed point theorems in intuitionistic fuzzy metric space[51].

1.10 FIXED POINT THEOREMS IN FUZZY 2-METRIC AND 3-METRIC SPACES

S. Gahlelr has examined the idea of 2-metric space in a number of works. Investigated contraction type mappings in 2-metric space for the first time. The investigation of probabilistic metric spaces was started by Z. Wenzhi and numerous others. For a pair and triplet of self-mappings on 2-metric spaces meeting contraction type criteria, later common fixed point theorems have been proven.

Fuzzy 2-metric space and fuzzy 3-metric space were first established in 2005 by Sushil Sharma[40], who also found certain common fixed point theorems for three mappings in this context. By proving common fixed point theorems for commuter maps, Sushil Sharma updated and expanded Fisher's findings. Amardeep Singh et al. were inspired by this and were able to discover common fixed point theorems for compatible maps in fuzzy 2-metric space.

We keep in mind that an object may or may not be fuzzy if the space between them is fuzzy. That is, the set will be fuzzy in fuzzy metric space, but the distance between items in terms of the nearness function will be fuzzy in fuzzy 2-metric space, and the set may or may not be fuzzy. The area function in Euclidian spaces first proposed the abstract features of 2-metric space, which is typically a real valued function of a part triples on a set X. The volume function suggests that 3-metric space is now what one would naturally anticipate.

1.11 FIXED POINT THEOREM IN RANDOM FUZZY METRIC SPACE

The concept of a fuzzy random variable, which is analogous to the idea of a random variable, was developed in order to apply statistical analysis to situations when the outcomes of a random experiment are ambiguous. But unlike conventional statistical techniques, no one definition had been developed previous to Volker's work. He developed the notion of a fuzzy random variable from the perspective of set theory, using the general topology approach and results from the theories of topological measure and analytic spaces. In fixed point, random fuzzy spaces do not introduce any outcomes.

1.12 A FIXED POINT THEOREM IN CONE METRIC SPACE

A famous issue in metric spaces is the investigation of fixed points for contractive mappings, and Huang and Zhang's introduction of the cone metric space is one such generalization. They gave some essential results for a self-map meeting a contractive condition in this space and replaced the set of real numbers from a metric space with an ordered Banach space.

This is a significant step in the development of cone metric space fixed point theory. Ali Abou Bakr, S,M, given their contribution in cone metric space fixed point theory[4].Verma, Kabir, Chauhan and shrivastava has generalized fixed point theorem for multi-valued contractive mapping in cone b-metric space[50]

1.13 COUPLED FIXED POINT IN TWO G- METRIC SPACE

Generalised metric space was first introduced in 2006 by Mustafa and Sims, who also provided several fixed point theorems in G-metric space.

V. Lakshmikantham developed the idea of a linked coincidence point of mapping. They also looked at a few fixed point theorems in partially ordered metric spaces. In generalised metric spaces, linked coincidence fixed point theorems in

2011.

The investigation of common fixed point theory in G-metric spaces was started by Abbas and Rhoades[24] in 2009.Recent common fixed point theorems were provided in two G-metric spaces in a fundamentally new and more organic method by Feng gu. In 2016, Rahim Shah, Akbar Zada, and Tongxing Li[28] presented the idea of integral type contraction regarding generalized metric space and demonstrated some novel common coupled coincidental fixed point results of integral type contractive mappings in generalized metric space. Latif, Nazir and Abbas presented the stability of fixed points in generalized metric spaces[27]. Pathak & Gharib, Malkawi, Rabaiah, Shatanawi and Alsauodi have given their contribution to a fixed point theorem in matric space[17,29]. Hong, Pasman, Quddus and Mannan has contribute to supporting risk management decision making by converting linguistic graded qualitative risk matrices through interval type-2 fuzzy sets[22]. Fixed point theorems for nonlinear contractive mappings inordered b-metric space with auxiliary function[38]. Ansari, Chandok, Hussain, Mustafa and Jaradat has proved some common fixed point theorems for weakly α-admissible pairs in G-metricspaceswith auxiliary function[5].

Banach space, Hilbert space, fuzzy set, fuzzy subset generated by mapping, fuzzy real numbers, fuzzy metric spaces, fuzzy normed linear spaces, fuzzy uniform spaces, fuzzy metric spaces with respect to continuous t-norm, fuzzy 2-metric spaces, fuzzy 3-metric spaces, two generalized metric spaces, random fuzzy metric spaces, cone metric spaces, and generalized metric spaces are some of the concepts that have been defined and the results that follow them.

1.14 BANACH SPACE

Over the past three decades, there has been a lot of research done on fixed point theorems and common fixed point theorems satisfying contractive type conditions. Polish mathematician Banach established a theorem in 1922 that guarantees the existence and uniqueness of a fixed point under approximation conditions. The Banach fixed point theorem or Banach contraction principle are two names for his conclusion. This theorem offers a method for resolving several applicable issues in engineering and the mathematical sciences. Numerous researchers have improved, expanded, and generalized.

Banach's fixed point theorem in different ways.

Definition 1.14.1: Let X be a linear space (= vector space) over the flied f of complex scalars. Then X is a normed linear space if for every $f \in X$ there is a real number $||f||$ called the norm of f such that:

a. $||f|| \geq 0,$

b. $||f|| = 0$ if *and only* if $f = 0,$

c. $||c\,f|| = ||c||\ ||f||$ for every scalar c,

d. $||f + g|| \leq ||f|| + ||g||.$

Definition 1.14.2: Let X be a normed space and let $\{f_n\}_{\in \rho}$ be a sequence of elements of X.

a. $\{f_n\}_{n \in \rho}$ Converges to $f \in X$ if $\lim_{n\to\infty} ||f - f_n|| = 0, \exists\, N > 0,\ \forall\ \varepsilon > 0, n \geq N, ||f - f_n|| < s$. In this case we write $lim_{n\to\infty} f_n = f \ or\ f_n \to f.$

b. $\{f_n\}_{n \in \rho}$ is Cauchy if $\forall\ \varepsilon > 0, \exists\, N > 0, \forall\, m, n \geq N, ||f - f_n|| < \varepsilon.$

Definition 1.14.3: Any convergent sequence in a normed linear space is easily demonstrated to be a Cauchy sequence. The statement that all Cauchy sequences converge in any normed linear space may or may not be accurate. A normed linear space X is considered complete if and only if it possesses the property that all Cauchy sequences converge. A Banach space is a fully normed linear space.

1.15 HILBERT SPACE

German mathematician David Hilbert (1862–1943), who made several contributions to the growth of mathematics, is best remembered for his groundbreaking work in the area of functional analysis. The mathematical formulation of quantum theory relies heavily on the notion of Hilbert space. David Hilbert, who developed the idea in the setting of integral equations, is the name given to these spaces.

Definition 1.15.1: Let H be a vector space. Then H is an inner product space if for every $f, g, \in X$ there exists a complex number (f, g) called the inner product of f and g such that:

(a) (f, f) is real and $(f, f) \geq 0.$

(b) $(f, f) = 0$ if and only if $f = 0$.

(c) $(g, f) = \overline{< f, g >}$

(d) $(af_1 + bf_2, g) = a(f_1, g) + b\,(f_2, g)$

Each inner product determines a norm by the formula $||f|| = (f, f)^{½}$. Hence every inner product space is a normed linear space. The Cauchy- Schwarz inequality states that $| < f, g > | \leq ||f||\ \ ||g||,\ \forall\ f, g\ \in H$.

A Hilbert space is one in which an inner product space H is complete. A Hilbert space is, in other words, a Banach space whose norm is defined by an inner product.

1.16 METRIC SPACE

Definition 1.16.1: Let X be any set. Let $d(x, y)$ be a function defined on the set X $\times$ X satisfying the following condition:

1. $d(x, y) \geq 0$
2. $d(x,\ y) = 0$ if and only if $x = y$
3. $d(x, y) \leq d(x, z) + d(z, y)$ *triangle inequality.*

Such a function $d(x, y)$ is called a metric space on X, it is a mapping of $X \times X \rightarrow R$. A set X with a metric d is called a metric space.

Definition 1.16.2: A Sequence of points $\{X_n\}$ is said to converge to a point x of X if $d\ (X_n, X) \rightarrow 0$ as $n \rightarrow \infty$ there exists an integer n_0 depends on ϵ, such that $0 \leq d(X_n, X) < \epsilon$, for each $n \geq n_0$. The point x is called the limit of the sequence.

Definition 1.16.3: A sequence $\{X_n\}$ in a metric space (X, d) is called a Cauchy sequence if $\forall\ \varepsilon > 0$ there exists an integer n_0 such that $d(x_n, x_m) < \varepsilon$ for each $m > n \geq n_0$.

Definition 1.16.4: Let f and g be self maps of a metric space (X, d) then f and g are said to commuting if and only if $fg = gf$.

Definition 1.16.5: Let f and g be self maps of a metric space (X, d). A point $x \in X$ is said to be a coincidence point of f and g if $fx = gx$

Definition 1.16.6: Two self-maps S and T of a metric space (X, d) are said to be compatible if $lim_{n\rightarrow\infty}\ d(STx_n, TSx_n) = 0$ whenever $\{X_n\}$ is a sequence in X such that $lim_{n\rightarrow\infty}\ Sx_n = lim_{n\rightarrow\infty}\ Tx_n = t$, for $t\ \in$ X.

1.17 FUZZY SET

Lotif. A. Zadeh was the first to propose the origins of the fuzzy set. The outcomes of fuzzy sets used in our research are presented here without justification. We consult the relevant sources for more information.

Definition 1.17.1: A function A from a non-empty set X into unit interval I is called a fuzzy set in X. $\forall x \in A, A(x)$ is called the grade of membership of x in A. A is also said to be a fuzzy subset of x.

Definition 1.17.2: A fuzzy subset A is said to be empty if $A(x) = 0$ for all x in X and A is whole if A(x) =1 for all x in X. The empty fuzzy set is denoted by Ø or 0 and whole set is denoted by X or 1.

Definition 1.17.3: Let X be a set and A and B be two fuzzy subsets of X. A is said to be included in B if $A(x) \leq B(x)$ for all $x \in$ X. It is denoted by $A \leq B$.

(a) A is said to be equal to B if A(x) = B (x) for all $x \in$ X and written as A =B.

(b) B is said to be complement of A if B (x) =1 - A (x) for every $x \in$ X and denoted by $B = A^c. Obviously\ (A^c)^C = A$.

Definition 1.17.4: The union of A and B be defined by $A \cup B\,(x) = \max\{A(x), B(x)\}$ or $A(x) \cup B(x)\ \forall\ x \in X$.

The intersection of A and B be defined by $A \cap B\ (x) = \min\{A(x), B(x)\}\ or\ A(x) \cup B(x)\ \forall\ x \in X$.

The difference of A and B is defined by $A \backslash B = A \cap B$.

In general, if I is an index set and $A = \{A_\alpha \backslash \alpha \in I\}$ is a family of fuzzy subsets of X, then their union UA_α is defined by $(UA_\alpha)(x) = \sup\ \{A_\alpha(x) \backslash \alpha \in I\}, x \in X$ and the intersection $\cap\ A_\alpha$ is defined by $(\cap\, A_\alpha\ (x) = \inf\ \{A_\alpha(x) \backslash \alpha \in I\}, x \in X$.

1.18 FUZZY REAL NUMBER

Definition 1.18.1: A fuzzy real number x is a fuzzy set on the real axis i.e. a mapping $x: R \rightarrow [0,1]$ associating with each real number t its grade of membership x(t).

Definition 1.18.2: A fuzzy real number x is convex if $x(t) \geq x(s) \wedge x(r) = \min\{x(s), x(r)\}$ where $s \leq t \leq r$.

Definition 1.18.3: The α -level set of a fuzzy real number x is denoted by $[x]\alpha = \{t: x(t) \geq t\}, 0 < \alpha \leq 1$.

A fuzzy real number x is called normal if $x(t_0) = 1$ for some $t_0 \in R$. x is called upper semi continuous if $\forall$ S > 0, $x^{-1}([0, \alpha + s]), \forall \alpha \in I$ is open in the usual topology of R. It can be easily seen that the $a - level\ set\ [x]_\alpha$ of an upper semi continuous, normal, convex fuzzy real number x for each $\alpha, 0 < \alpha \leq 1$ is closed interval $[\alpha^\alpha, b^\alpha]$ where $\alpha^\alpha = -\infty\ and\ b^\alpha = \infty$ are also admissible. The set of all upper semi continuous, normal, convex fuzzy real number is denoted by E or R (I). Since each $r \in R$ can be considered as a fuzzy real number $\bar{r}$, denoted by $r(t) = 1, \{t = 0, t \neq r\ then\ \bar{r} \in E\}$. in other words R can be embedded in E or R(I).

Definition 1.18.4: A fuzzy real number x is called non negative if $x(t) = 0$ for all $t < 0$.

The set of all non negative fuzzy real number is denoted by G or $R^*(I)$.

The equality of fuzzy real number x and y is defined by $x(t) = y(t)$ for all $t \in R$.

Definition 1.18.5: For $k \in E, k_x(t) = x\ (k^{-1}t)$ and $0x$ is defined to be $\bar{0}$.

Definition 1.18.6: A partial ordering $' \leq'\ in\ E$ is defined by $x \leq y$ if and only if

$a_1^\alpha \leq a_2^\alpha$ and $b_1^\alpha \leq b_2^\alpha$ for all $\alpha \in (0,1)$, where $[x] = [a_1^\alpha, b_1^\alpha]$ and $[y]\alpha = [a_2^\alpha, b_2^\alpha]$.

Definition 1.18.7: A sequence $\{x_n\}$ in E converges to X $\in$ E, denoted by $lim_{n\to\infty}$ $X_n = x$ if $lim_{n\to\infty}\ a_n^\alpha = lim_{n\to\infty}\ b_n^\alpha = b^\alpha\ \forall\ \alpha \in (0,1)$, where $[X_n]\ \alpha = [a_n^\alpha, b_n^\alpha]$ and $[X]\ \alpha = [a^\alpha, b^\alpha]$.

1.19 FUZZY METRIC SPACE

Definition 1.19.1: Let X be a non empty set, d a mapping from X×X into G or $R^*(I)$ and let the mappings $L, R: [0,1] \times [0,1] \to [0,1]$ be symmetric, non decreasing in both arguments and satisfy $L\ (0,0) = 0$ and $R\ (1,1) = 1$ Denoted $[d(x,y)]\ \alpha = [\lambda_\alpha\ (x,y),\ p_\alpha\ (x,y)]$ for $x,\ y \in X, 0 < \alpha \leq 1$.

Remark: λ_α(x, y) is called left end point and ρ_α(x, y) is right end point of α-level set of d(x,y).

The quadruple (X, d, L, R) is called a fuzzy metric space and d, a fuzzy metric if

i. $d(x,\ y) = 0$ if and only if $x = y$.

ii. $d(x, y) = d(y, x)$ for all $x, y, \in$ X.

iii. $\forall\, x,\ y, z \in$ X

a. $d(x, y)(s + t) \geq L\big(d(x, z)(s),\ d(z, y)(t)\big)$ whenever $s \leq \lambda_1(x, z), t \leq \lambda_1\ (z, y)$ and $s + t \leq \lambda_1(x, y)$

b. $d(x, y)(s + t) \geq R\big(d(x, z)(s), d(z, y)(t)\big)$ whenever $s \leq \lambda_1(x, z), t \leq \lambda_1(z, y)$ and $s + t\ \leq\ \lambda_1(x, y)$

The usual metric space is special case of the fuzzy metric space.

Lemma 1.19.1: The triangle inequality (iii) (b) with R=Max is equivalent to the triangle inequality $p_\alpha(x, y) \leq\ p_\alpha(x, z) +\ p_\alpha(z, y)$for all α ∈ [0,1] and x , y, z ϵ X.

Lemma 1.19.2: The triangle inequality (iii) (a) with L=Min is equivalent to the triangle inequality $\lambda_\alpha(x, y)\ \leq\ \lambda_\alpha\ (x, z) + \lambda_\alpha\ (z, y)$for all α ϵ (0,1) and x, y, z ϵ X.

Theorem 1.19.1: In a fuzzy metric space (X, d, Min, Max) the triangle inequality is equivalent $d(x, y) \leq d(x, z) + d(z, y)$ for x, y, z ϵ X.

Definition 1.19.2: Let (X, d, L, R) be a fuzzy metric space. A sequence $\{x_n\}$ is said to be convergent to $x \in$ X if and only if $lim_{n\to\infty}(x_n, x) =\ \bar{0}$ and denoted by $lim_{n\to\infty}\ x_n =\ x$.

Lemma 1.19.3: A sequence $\{x_n\}$ in a fuzzy metric space (X, d, L, R) converges to $x \in$ X if and only if $lim_{n\to\infty}(x_n, x) = 0, \alpha\ \in\ (0,1)$.

Definition 1.19.3: A sequence $\{x_n\}$ in a fuzzy metric space (X, d, L, R) is called a Cauchy sequence if $lim_{m,n\to\infty} d\ (x_m, x_n) =\ \bar{0}$.

Lemma 1.19.4: A sequence $\{x_n\}$ in a fuzzy metric space (X, d, L, R) is called a Cauchy sequence if $lim_{m,n\to\infty}\ p_\alpha\ (x_m, x_n) = 0$.

Definition 1.19.4: If every Cauchy sequence in a fuzzy metric space X converges to a point in X, then X is said to be complete.

1.20 FUZZY NORMED LINEAR SPACE

Definition 1.20.1: Let X be a vector space over R. Let $||.||$: X $\to E$ and let the mapping $L, R: [0,1] \times [0,1] \to [0,1]$ be symmetric, non decreasing in both argument and satisfy L(0, 0) = 0 and R(1,1)=1.

We write $[||x||]\alpha = [|| \, |x| \, ||^{\alpha}, \; || \, |x| \, ||^{\alpha}]$ for $x \in X, 0 < \alpha \leq 1$ and suppose for all $x \in X, x \neq 0$

there exists $a_0 \in X, [0,1]$ independent of x such that for all $a \leq a_0$

a. $|| \, |x|^2 \, ||^{a} < \infty$

b. $\inf || \, |x|^1 \, ||^{a} > 0$

The quadruple $(X, ||.||, L, R)$ fuzzy norm, if

(i) $||x|| = \bar{0}$ if and only if $x = 0$.

(ii) $||rx|| = |r| \; ||x||, x \in X, r \in R$.

(iii) $\forall x, y \in X$,

a. Whenever $s \leq \forall || \, |x| \, ||_1^1, t \leq || \, |y| \, ||_1^1$ and $s + t \leq || \, |x + y| \, ||_1^1$ $||x + y|| \; (s + t) \geq L \, (||x|| \, (s), ||x|| \, (t))$

b. Whenever $s \geq || \, |x| \, ||_1^1, t \geq || \, |y| \, ||_1^1$ and $s + t \geq || \, |x + y| \, ||_1^1$ $||x + y|| \, (s + t) \leq R \, (||x|| \, (s), ||x|| \, (t))$

Definition 1.20.2: Let $(X, ||.||)$ is a fuzzy normed linear space. A sequence $\{X_n\} \in X$ is said to converge to $x \in X$, denoted by $lim_{n\to\infty} x_n = x$ if and only if $lim_{n\to\infty} \|x_n - x\| = \bar{0}$, i.e. $lim_{n\to\infty} || \, |x_n - x| \, ||_1^{\alpha} = lim_{n\to\infty} || \, |x_n| \, ||_2^{\alpha} = 0 \; \forall \alpha \in (0,1)$.

Definition 1.20.3: A sequence $\{X_n\}$ in a fuzzy normed linear space $(X, ||.||)$ is called Cauchy sequence if $lim_{n\to\infty} \|x_m - x_n\| = \bar{0}$. $i.e.$ if $lim_{m,n\to\infty} || \, |x_m - x_n| \, ||^{\alpha} = \bar{0}, \forall \alpha \in (0,1)$.

1.21 FUZZY UNIFORM SPACE

In function analysis, uniform spaces lie between metric space and general topological space.The same concept applies in fuzzy structure.

Definition 1.21.1: A uniform space X is sequentially complete if each Cauchy sequence in X converges in X.

Definition 1.21.2: Let (X, d, L, R) be a fuzzy metric space with $lim_{a\to 0^+} R(a, a) = 0$. If (X, d, L, R) is a complete fuzzy metric space then (X, V) is a sequentially complete uniform space.

Result 1.21.3: Family of fuzzy metrics induced by fuzzy metric space (X, d, L, R): In a fuzzy metric space (X, d, L, R) the family $\{p_\alpha : \alpha \in (0,1)\}$ satisfies the following condition:

i. for all $\alpha \in (0,1)$and $x \in X,\ p_\alpha(x,x) = 0$;

ii. for all $\alpha \in (0,1\}$ and $x, y \in X, p_\alpha\,(x,y) = p_\alpha\,(y,x)$;

iii. if $p_\alpha\,(x,y) = 0$ for all $\alpha \in (0,1)$, then $x = y$.

If R=Max, then p_α also satisfies $p_\alpha(a,y) \le p_\alpha(x,z) + p_\alpha(z,y)$

If $lim_{t\to\infty}\ d(x,y)\ (t) = 0, \forall\ x,\ y \in X$, then $p_\alpha(x,y) < \infty, \forall\ \alpha \in (0,1)$ and x, y .

Hence if (X, d, L, R) is a fuzzy metric space with $lim_{t\to\infty}\ d\ (x,y)(t) = 0\ \forall x,y \in X$

then the family p_α (x, y) $= 0, \forall\ \alpha \in (0,1)$, then $x = y$.

Theorem 1.21.4: if (X, d, L, Max) is a fuzzy metric space then for any $\alpha \in (0,1)$

i. $u(\varepsilon_1, \alpha) \le u\ (\varepsilon_2, \alpha) for\ 0 \le \varepsilon_1 \le \varepsilon_2\ and$

ii. $u(\varepsilon_1, \alpha)\ .\ u\ (\varepsilon_2, \alpha) \le u\ (\varepsilon_1 + \varepsilon_2\ \alpha)\ for\ any\ E_1 E_2 > 0$

where $\mathrm{u}\ (\varepsilon_1, \alpha).\mathrm{u}(\varepsilon_2, \alpha) = \{(\mathrm{x},\mathrm{y}) : \varepsilon z \in \mathrm{X}$ with $(\mathrm{x},\mathrm{z}) \in \mathrm{u}\ (\varepsilon_1, \alpha)\ and\ (\mathrm{z},\mathrm{y}) \in \mathrm{u}\ (\varepsilon_2, \alpha)\}$

1.22 FUZZY METRIC SPACE WITH RESPECT TO CONTINUOUS T-NORM

Definition 1.22.1: A binary operation *: [0, 1] × [0, 1] → [0, 1] is said to be continuous t-normif it satisfies the following condition:

i. * is commutative and associative;

ii. * is continuous;

iii. $a * 1 = a$ for $a \in [0,1]$.

iv. $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Examples of continuous t-norms are Lukasiewicz t-norm, that is, $a_L^+ b = \max\{a + b - 1, 0\}$, product t-norm, that is, $a_P^*\ b = ab, and\ minimum\ t - norm, that\ is, a_M^* b = \min\{a, b\}$.

Example 1.22.1: Define $a * b = \min\ (a, b), \forall\ a,\ b\ \in [0,1].\ then *$ $is\ a\ continuous\ t - norm.$

Remark 1.22.1 Given an arbitrary set X, a fuzzy set M on X is function from X to the unitinterval $[0,1]. Let\ [0,1]^{X} = \{f : X \to [0,1]\}$ thus $M\ \in [0,1]^{X}$.

Definition 1.22.2: A fuzzy metric space is a triple (X; M; *), where X is a non-empty set, * is acontinuous t-norm and M is a fuzzy set on $X \times X \times [0, +\infty]$, satisfying the following properties:

1. $M\ (x,\ y,\ 0) = 0$ for all $x; y\ \in X$;
2. $M\ (x,\ y,\ t) = 1$ for all $t > 0$ if f $x = y$;
3. $M\ (x,\ y,\ t) = 1$ for all $t > 0$ if f $x = y$;
4. $M\ (x,\ y,\ \cdot): [0, +\infty] \to [0, 1]$ is left continuous for all $x, y\ \in X$;
5. $M\ (x,\ z,\ t + s)\ \geq M(x,\ y,\ t) * M(y, z, s)$ for all $x,\ y,\ z\ \in X$ and for all $t,\ s > 0$;

Remark 1.22.2: Definition 1.16.4 (4) means that for each x, y ∈ X there is a function $M_{xy}: [0, \infty] \to I, t\ \to M(x, y, t), M\ (x, y, t)$ Can be notion of as the

degree of nearness between a and y with respect to $t \geq 0$.

We can fuzzify example of metric spaces into fuzzy metric space in a natural way:

Example 1.22.2: Let (X, d) be a metric space define $a * b = \min\ \{a, b\}\ \forall\ a,\ b\ \in X,$ we have $M(x, y, t) = \frac{t}{t = d(x,y)}$ for all x, y in X and t > 0. Then (X, M, *) is fuzzy metric induced by a metric d is called the standard fuzzy metric space.

The definition that follows is an adjustment to Definition 1.16.4. This adjustment is required since the fuzzy metric in Definition 1.16.4 does not produce a Hausdorff topology.

Definition 1.22.3: The 3-tuple (X, M,*) is called a fuzzy metric space (shortly FM-space) if X is an arbitrary set * is a continuous t-norm and M is a fuzzy set in $X^2 \times$ $[0, \infty]$ satisfying the following conditions: for all x, y, z in X and s, t >0,

i. M(x, y, t) > 0;

ii. M(x, y, t)=1, for all t > 0 if f x=y,

iii. M(x, y, t)=M(y, x, t),

iv. M(x, y, t)*M(y, z, s) ≤ M(x, z, t+s)

v. $M(x, y): [0, \infty) \to [0,1]$ is left continuous.

vi. $\lim_{t\to\infty} M(x, y, t) = 1$ For all x, y in X.

In the sequel the fuzzy set M as in Definition 1.16.7 will be referred to as a fuzzy metric.It shall be shown that the topology induced by the fuzzy metric space (X, M, *) is Hausdorff.

Lemma 1.22.1: M(x, y,.) is non decreasing for all x, y in X.

Definition 1.22.4: A sequence $\{X_n\}$ is a fuzzy metric space (X,M,*) is said to converge to $x \in X$ if for each $s, 0 < s < 1$ and $t > 0, n_o \in N$ such that

$M(x_n, x, t) > 1 - s, \forall\, n \geq n_0$.

Definition 1.22.5: A sequence $\{X_n\}$ is a fuzzy metric space (X, M, *) is said to be a Cauchy sequence if for each $s, 0 < s < 1$ *and* $t > 0,\ n_0 \in N$ *such that* M (x_n, x_m, t) >1-ε, ∀ n, m ≥ n_0.

Definition 1.22.6: A fuzzy metric space is said to be complete if every Cauchy sequence isconvergent.

Definition 1.22.7: A function M is continuous in fuzzy metric space if $x_n X, y_n \to y$ then $\lim_{n\to\infty} M(x_n,\ y_n,\ t) = M(x, y, t)$ for $t > 0$.

1.23 TOPOLOGY AND FUZZY METRIC SPACES

We continue to present some concept and results from classical metric spaces theory in thecontext of fuzzy metric space.

Definition 1.23.1: Let (X, M, *) be a fuzzy metric space. We define the open ball B (x, r, t) with centre $x \in X$ and radius $r, 0 < r < 1, t > 0$, as B $(x,\ r,\ t) = \{y \in X: M(x, y, t) > 1 - r\}$

Definition 1.23.2: A subset A of a fuzzy metric space (X, M, *) is said to be open if given any point $a \in A$, there exists $0 < r < 1, t > 0$ such that $B(a, r, t) \leq A$.

Theorem 1.23.1: Every open ball in a fuzzy metric space (X, M, *) is an open set.

Theorem 1.23.2: Every fuzzy metric space is Hausdorff.

Preposition 1.23.1: Let (X, d) be a metric space and $M_d(x, y, t) = \frac{t}{t{=}d(x,y)}$

Corresponding standard fuzzy metric space on X. Then the topology T_d induced by the metric d and thetopology T_{Md} induced by the fuzzy metric space M_d are the same that is $T_d = T_{Md}$.

Definition 1.23.3: Let (X, M, *) be a fuzzy metric space. A subset A of X is said to be F-bounded if there exists t > 0 and 0 < r < 1 such that $M(x, y, t) > 1 - r$ for all $x, y, \in A$.

Remark 1.23.1: Let (X, M, *) be a fuzzy metric space induced by a metric d on X. then $A \leq X$ is F-bounded if and only if it is bounded. This is what R lower would call a goodextension of the notion of bounded.

Theorem 1.23.3: Every compact subset A of a fuzzy metric space X is F- bounded.

Remark 1.23.2: In a fuzzy metric space every compact subset is closed and bounded.

Theorem 1.23.4: Let (X, M, *) be a fuzzy metric space and T_M be the topology induced bythe fuzzy metric. Then for a sequence $\{X_n\}$ in X, the sequence $\{X_n\}$ converges to x if and only if (X_n, X, t) converges to 1 as n tends to ∞.

Remark 1.23.3: Let (X, M, *) be a fuzzy metric space and $\{X_n\}$ be a sequence in X. Then $lim_{n\to\infty} d(x,\ y) = 0$ if and only if $lim_{n\to\infty} M_d\ (x_n,\ x,\ t) = 1\ for\ all\ t > 0\ and\ x\ \in X.$

Definition 1.23.4: Let (X, M, *) be a fuzzy metric space. We define a closedball $B[x, r, t]$ with center $x \in X$ and radius $r, 0 < r < 1,\ t > 0\ \ as\ B[x, r, t] = \{y\ \in X: M(x, y, t) \geq 1 - r\}$

Lemma 1.23.1: Every closed ball in a fuzzy metric space (X, M, *) is closed set.

Theorem 1.23.5: Let (X, M, *) be a complete fuzzy metric space. Then the intersection of acountable number of dense open set is dense.

1.24 FUZZY 2-METRIC SPACE

Here we recall the definition of fuzzy 2-metric space.

Definition 1.24.1: An operation $*: [0,1]^3 \to [0,1]$ is called a continuous t-norm if $([0,1]\ *)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$ for all $a_1, a_2, b_1, b_2\ and\ c_1, c_2$ are in [0,1].

Definition 1.24.2: The triple (X, M, *) is called a fuzzy 2-metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^3 \times [0,1]$ satisfying the following condition:

i. $M(x, y, z, 0) = 0$

ii. $M(x, y, z, t) = 1; t > 0$ and when at least two of the three points are equal.

iii. $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$ (symmetry about three variables)

iv. $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) + M(x, u, z, t_2) + M(u, y, z, t_3)$ (This corresponds to tetrahedron inequality in 2-metric space).

v. $M(x, y, z, .): [0, \infty] \rightarrow [0,1]$ *is left continuous*; $\forall x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$.

Note 1.24.1: The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle formed by the three points x, y, z < t.

Definition 1.24.3: A function M is continuous in fuzzy 2-metric space if $x_n \rightarrow x, y_n \rightarrow y$ then $lim_{n\rightarrow\infty}(x_n, y_n, a, t) = M(x, y, a, t)$ for all $a \in X$ *and* $t > 0$.

Lemma 1.24.1: Let (X, M, *) be a fuzzy 2-metric space. Then M(x, y, z, ·) is non-decreasing function for all $x, y, z \in X$.

Definition 1.24.4: A sequence $\{x_n\}$ in a fuzzy 2-metric space (X, M, *) converge to a point x in X if and only if $lim_{n\rightarrow\infty} M(x_n, x, a, t) = 1$, for all $a \in X$ *and* $t > 0$.

Definition 1.24.5: let (X, M, *) be a fuzzy 2-metric space. A sequence $\{x_n\}$ is called a Cauchy sequence, if and only if $lim_{n\rightarrow\infty} M(x_{n+p}, x_n, a, t) = 1$, for all $a \in X$ *and* $t > 0, p > 0$.

Definition 1.24.6: A fuzzy 2-metric space (X, M, *) is said to be complete if and only if every Cauchy sequence in X converges in X.

Note 1.24.2: Let (X, M, *) be a fuzzy 2-metric space, then $lim_{n\rightarrow\infty} M(x, y, z, t) = 1$.

Lemma 1.24.2: If for all $x, y, z \in X, t > 0$ and $0 < k < 1, M(x, y, z, kt) \geq M(x, y, z, t)$ then $x = y$.

1.25 FUZZY 3-METRIC SPACE

Definition 1.25.1: A operation $*: [0,1]^4 \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 * d_1 \leq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2 \forall a_1, a_2, b_1, b_2, c_1, c_2,$ $and\ d_1, d_2\ are\ in\ [0,1]$.

Definition 1.25.2: The 3-tuple (X, M, *) is called a fuzzy 3-metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^4 \times [0,1]$ satisfying the following condition:

i. $\mathrm{M}\,(\mathrm{x,y,z,w,0}) = 0$

ii. $\mathrm{M}\,(\mathrm{x,y,z,w,t}) = 1; \mathrm{t} > 0$

iii. $\mathrm{M}\,(\mathrm{x,y,z,w,t}) = \mathrm{M}\,(\mathrm{x,w,z,y,t}) = \mathrm{M}\,(\mathrm{y,z,w,x,t}) = \mathrm{M}\,(\mathrm{z,w,x,y,t}) = \cdots$

iv. $M(x,y,z,w,t_1 + t_2 + t_3 + t_4) \geq M(x,y,z,u,t_1) * M(x,y,u,w,t_2) * M(x,u,z,w,t_3) * M(u,y,z,w,t_4)$

v. $M(x,\ y,\ z,\ w,\ .): [0,\infty) \to [0,1]\ is\ left\ continuous; for\ all$ $x,y,z,u,w\ \in X\ and\ t_1,t_2,t_3,t_4 > 0.$

Definition 1.25.3: A sequence $\{x_n\}$ in a fuzzy 3-metric space (X, M, *) converge to a point x in X if and only if $lim_{n\to\infty} M(x_n, x, a, b, t) = 1$, for all $a, b, \in X$ and $t > 0, p > 0$.

Definition 1.25.4: Let (X, M, *) be a fuzzy 3-metric space. A sequence $\{x_n\}$ is called a Cauchy sequence if f $lim_{n\to\infty} M\big(x_{n+p}, x_n, a,\ b,\ t\big) = 1$, for all $a, b, \in X$ and $t > 0, p > 0$.

Definition 1.25.5: A fuzzy 3-metric space (X, M, *) is said to be complete and only if every Cauchy sequence in X converges in X.

Definition 1.25.6: A function M is continuous in fuzzy 3-metric space if $x_n \to x, y_n \to y$ then $lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t)$ for all $a, b, \in X$ and $t > 0$.

❒❒❒

REFERENCES

[1.] C. T. Aage, B. S. Choudhury & K. Das. Some Fixed Point Results in Fuzzy Metric Spaces Using A Control Function. *Surveys in Mathematics & Its Applications*, 2017,*12*.

[2.] M. Abbas, V. Rakočević, & A. Iqbal. Fixed points of Perov type contractive mappings on the set endowed with a graphic structure. *Revista de La Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 2018, *112*, 209–228.

[3.] A. A. N. Abdou. Common fixed point results for compatible-type mappings in multiplicative metric spaces. *J. Nonlinear Sci. Appl*, *9*, 2016, 2244–2257.

[4.] Ali Abou Bakr, S. M. (2020). Theta cone metric spaces and some fixed point theorems. *Journal of Mathematics*, *2020*, 1–7.

[5.] A. H. Ansari, S. Chandok, N. Hussain, Z. Mustafa, & M. M. M. Jaradat. Some common fixed point theorems for weakly α-admissible pairs in G-metric spaces with auxiliary functions. *J. Math. Anal*, *8*(3), 80–107, 2017.

[6.] V. Brattka, S. Le Roux, J. S. Miller, & A. Pauly. Connected choice and the Brouwer fixed point theorem. *Journal of Mathematical Logic*, 2019,*19*(01), 1950004.

[7.] A. Bucur. About applications of the fixed point theory. *Scientific Bulletin*, 2017,*22*(1), 13–17.

[8.] A. M. Burton, Kramer, R. S. S., Ritchie, K. L., & Jenkins, R.. Identity from variation: Representations of faces derived from multiple instances. *Cognitive Science*, 2016, *40*(1), 202–223.

[9.] Y. J. Cho. Survey on metric fixed point theory and applications. *Advances in Real and Complex Analysis with Applications*, 2017, 183–241.

[10.] M. Cvetković. On the equivalence between Perov fixed point theorem and Banach contraction principle. *Filomat*, 2017, *31*(11), 3137–3146.

[11.] C. Daskalakis, C. Tzamos, & M. Zampetakis. A converse to Banach's fixed point theorem and its CLS-completeness. *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, 2018, 44–50.

[12.] P. Debnath, Z. D. Mitrović, & S. Y. Cho. Common fixed points of Kannan, Chatterjea and Reich type pairs of self-maps in a complete metric space. *São Paulo*

Journal of Mathematical Sciences, 2021, *15*, 383–391.

[13.] A. Dilo, R. A. De By, & A. Stein. A system of types and operators for handling vague spatial objects. *International Journal of Geographical Information Science*, 2007, *21*(4), 397–426.

[14.] K. K. Dompere. Fuzziness, Rationality, Optimality and Equilibrium in Decision and Economic Theories. *Fuzzy Optimization: Recent Advances and Applications*, 2010, 3–32.

[15.] H. Fatima. Fixed Point Theory in Hyper Convex Metric Spaces. *Elementary Education Online*, 2021, *20*(3), 1997.

[16.] G. Geometry. Fixed Point Theory. *Seminar of Mathematical Analysis: Proceedings, Universities of Malaga and Seville (Spain), September 2002-February 2003*, *64*, 195.

[17.] G. M. Gharib, A. A. Malkawi - A. M. Rabaiah, W. A. Shatanawi, & M. S. Alsauodi. A Common Fixed Point Theorem in An M*-Metric Space And An Application. *Nonlinear Functional Analysis and Applications*, 2022,*27*(2), 289–308.

[18.] M. E. Gordji, M. Ramezani, M. De La Sen, & Y. J. Cho. On orthogonal sets and Banach fixed point theorem. *Fixed Point Theory*, 2017a,*18*(2), 569–578.

[19.] M. E. Gordji, M. Ramezani, M. De La Sen, & Y. J. Cho. On orthogonal sets and Banach fixed point theorem. *Fixed Point Theory*, 2017b, *18*(2), 569–578.

[20.] S. Grecova, Sostak, A. P., & Uljane, I.. A construction of a fuzzy topology from a strong fuzzy metric. *Applied General Topology*, 2016, *17*(2), 105–116.

[21.] H. A. Hammad, & La Sen, M. De.. A technique of tripled coincidence points for solving a system of nonlinear integral equations in POCML spaces. *Journal of Inequalities and Applications*, *2020*(1), 1–27.

[22.] Y. Hong, H. J. Pasman, N. Quddus, & M. S. Mannan. Supporting risk management decision making by converting linguistic graded qualitative risk matrices through interval type-2 fuzzy sets. *Process Safety and Environmental Protection*, 2020,*134*, 308–322.

[23.] S. Khalehoghli, H. Rahimi, & M. E. Gordji. Fixed point theorems in R-metric spaces with applications. *AIMS Mathematics*, 2020, *5*(4), 3125–3137.

[24.] S. H. Khan, M. Abbas, & T. Nazir (n.d.). *Research Article Fixed Point Results for Generalized Chatterjea Type Contractive Conditions in Partially Ordered G-Metric Spaces*.

[25.] K. Kish Bar-On. *Connecting the revolutionary with the conventional: Rethinking the differences between the works of Brouwer, Heyting, and Weyl,*2022.

[26.] M. V. Kumar & S. M. Subhani. Application of common fixed point theorem on fuzzy metric space. *Proceedings of the World Congress on Engineering and Computer Science*, 2017, *1*.

[27.] A. Latif, T. Nazir & M. Abbas. Stability of fixed points in generalized metric spaces. *Journal of Nonlinear and Variational Analysis*, 2018, *2*(3), 287–294.

[28.] S. Li, A. Zada, R. Shah & T. Li . Fixed point theorems in dislocated quasi-metric spaces. *J. Nonlinear Sci. Appl*, 2017,*10*, 4695–4703.

[29.] H. K. Pathak & H. K. Pathak. Fixed Point Theorems. *An Introduction to Nonlinear Analysis and Fixed Point Theory*, 2018,231–448.

[30.] V. Parvaneh, N. Hussain, A. Mukheimer & H. Aydi. On fixed point results for modified JS-contractions with applications. *Axioms*, 2019a, *8*(3). https://doi.org/10.3390/axioms8030084

[31.] V. Parvaneh, N. Hussain, A. Mukheimer & H. Aydi. On fixed point results for modified JS-contractions with applications. *Axioms*, 2019b, *8*(3). https://doi.org/10.3390/axioms8030084

[32.] A. Pourmoslemi, S. Rezaei, T. Nazari & M. Salimi. Generalizations of Kannan and Reich Fixed Point Theorems, Using Sequentially Convergent Mappings and Subadditive Altering Distance Functions. *Mathematics*, 2020, *8*(9), 1432

[33.] M. Ren, X. Huang, X. Zhu & L. Shao. Optimized PSO algorithm based on the simplicial algorithm of fixed point theory. *Applied Intelligence*, 2020,*50*, 2009–2024.

[34.] R. M. Rodríguez, B. Bedregal, H. Bustince, Y. C. Dong, B. Farhadinia, C. Kahraman, L. Martínez, V. Torra, Y. J. Xu, & Z. S. Xu. A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress. *Information Fusion*, 2016, *29*, 89–97.

[35.] R. I. Sabri, M. Rasheed, O. Alabdali, S. Shihab & T. Rashid. On Some Properties in Fuzzy Metric Space. *Journal of Al-Qadisiyah for Computer Science and Mathematics*, 2021, *13*(1), Page-55.

[36.] K. Sawangsup & W. Sintunavarat. Fixed point results for orthogonal Z-contraction mappings in O-complete metric space. *Int. J. Appl. Phys. Math*, 2020,*10*(1), 33–40.

[37.] S. Sedghi, A. Gholidahneh & K. P. R. Rao. Common fixed point of two R-weakly commuting mappings in Sb-metric spaces. *Math. Sci. Lett*, 2017, *6*(3), 249–253.

[38.] N. Seshagiri Rao, K. Kalyani, & B. Mitiku. Fixed point theorems for nonlinear contractive mappings in ordered b-metric space with auxiliary function. *BMC Research Notes*, 2020,*13*(1), 1–8.

[39.] J. Shalf. The future of computing beyond Moore's Law. In *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* (Vol. 378, Issue 2166). Royal Society Publishing. 2020, https://doi.org/10.1098/rsta.2019.0061

[40.] S. Sharma & J. K. Tiwari. Common fixed point in fuzzy metric spaces. *The Pure and Applied Mathematics*, 2005, *12*(1), 17–31.

[41.] M. Shulman. Brouwer's fixed-point theorem in real-cohesive homotopy type theory. *Mathematical Structures in Computer Science*, 2018a, *28*(6), 856–941.

[42.] M. Shulman. Brouwer's fixed-point theorem in real-cohesive homotopy type theory. *Mathematical Structures in Computer Science*, 2018b, *28*(6), 856–941.

[43.] S. Shukla, S. Balasubramanian & M. Pavlović. A generalized Banach fixed point theorem. *Bulletin of the Malaysian Mathematical Sciences Society*, 2016*3,9*, 1529–1539.

[44.] P. Shamoi, A. Inoue & H. Kawanak. Fuzzy model for human color perception and its application in e-commerce. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 2016, *24*, 47–70.

[45.] I. Skalna, B. Rebiasz, B. Gawel, B. Basiura, J. Duda, J. Opila, & T. Pelech-Pilichowski. Advances in fuzzy decision making. *Studies in Fuzziness and Soft Computing*, 2015, *333*.

[46.] A. Turab, & W. Sintunavara. On the solution of the traumatic avoidance learning model approached by the Banach fixed point theorem. *Journal of Fixed Point Theory and Applications*, 2020,*22*, 1–12.

[47.] N. Tsuchiya, S. Taguchi & H. Saig. Using category theory to assess the relationship between consciousness and integrated information theory. *Neuroscience Research*, 2016, *107*, 1–7.

[48.] N. Van Dung & A. Petruşe. On iterated function systems consisting of Kannan maps, Reich maps, Chatterjea type maps, and related results. *Journal of Fixed Point Theory and Applications*, 2017,*19*, 2271–2285.

[49.] C. Vizman. Central extensions of semidirect products and geodesic equations. *Physics Letters A*, 2004,*330*(6), 460–469.

[50.] R. Verma, Q. A. Kabir, M. S. Chauhan, & R. Shrivastava. Generalized Fixed point Theorem for multi-valued contractive mapping in cone b-metric space. *Advances and Applications in Mathematical Sciences*, 2020,*19*(8), 825–832.

[51.] N. Yadav, P. K. Tripathi & S. L. Maurya. Fixed-point theorems in intuitionistic fuzzy metric space. *J. Math. Comput. Sci.*, 2021,*11*(2), 1305–1311.

[52.] Z. Zhang, J. Gao, Y. Gao & W. Yu. Two-sided matching decision making with multi-granular hesitant fuzzy linguistic term sets and incomplete criteria weight information. *Expert Systems with Applications*, 2021,*168*, 114311.

❐❐❐

CHAPTER – II

REVIEW OF LITERATURE

CONTENTS

The literature related with the subject entitled "Common Fixed Points of Compatible Maps in Fuzzy Metric Spaces and Fuzzy Mathematics" is presented in this chapter has been classified into three different sections namely Fixed Points Theory and Its Application, Common Fixed Points Application for Compatible Maps and Fuzzy Metric Space and Common Fixed Point to present the research works done in the subject area in readable and synchronized format.

2.1 FIXED POINTS THEORY AND ITS APPLICATION

- Researchers have been fascinated by fixed point theory ever since Banach's famous fixed point theorem was published in 1922[1]. The literature that is now available demonstrates that Fixed Point Theory has always been an active area of study. If x = x, then a self-map of a metric space X is said to have a fixed point. In addition to serving as tools for demonstrating the existence and uniqueness of solutions for various mathematical models representing phenomena arising in various fields of study, such as steady-state temperature distribution, fluid flow, chemical equations, economic theories, epidemiology, etc., fixed point theorems are also related to the existence and properties of fixed points. It is also used to research issues with systems that are relevant to optimum control. A subfield of mathematics known as fixed point theory looks for any self-mappings in which among all the elements one of the element leftward invariant. Brouwer's fixed point theorem in 1912 served as the foundation for topological fixed point theory. Tarski's fixed point theorem from 1955 is the source of discrete fixed point theory. It is the most commonly used methodology in non-linear analysis, and the effectiveness of fixed point theory is proved in the topological and algebraic structures[2]. The availability and distinct characteristics of the fixed point theory for self-maps over the metric space could better be resolute through the changing the distances in between the points, particularly when the points are working over the control functions[3].
- The theory of metric space in mathematics uses the Banach fixed point theorem[1], sometimes referred to as the contraction mapping theorem or contraction map0ping principle. It offers a contractive approach to locate certain fixed points and ensures their existence and uniqueness for particular self-mappings of metric space. The theorem, which Stefan Banach (1892–1945) initially formulated in 1922. Functional analysis[4–7] examines linear functions that appropriately trails the

vector spaces with limit-related structures. The study of the formulation characteristics of transformations, operators between function spaces, and the spaces of functions[8–9] can be linked to the beginnings of this field. The use of integral and differential equations illustrates the significance of this field of research[10–14].

- A number of fixed point theorems in convex b-metric spaces and their applications were researched by Lili Chen[15] et al. in 2020. Research claimed that the framework of b-metric spaces showed a technique of generalising the Mann's iteration algorithm and a number of fixed point solutions. First, a convex structure is used to establish the idea of a convex b-metric space, and Mann's iteration technique is then expanded to include this space. The strong convergence theorems for two categories of contraction mappings in convex b-metric spaces are then established with the aid of Mann's iteration technique. Additionally, for the aforementioned mappings in full convex b-metric spaces, the T-stability issues of Mann's iteration process are obtained.
- Fisher[16] demonstrated the fixed point theorem using several metric spaces and an increasing function from R+ to R+. Following that, several mathematicians who study in this field use various mapping techniques, including non-expansive mappings, self-mapping, multivalued mapping, sequences of mappings, operators in Hilbert spaces, and other mapping in metric spaces, Hilbert spaces, and Banach spaces. To determine the existence and uniqueness of a solution to higher order differential and integral equations, the fixed point theorem or contraction mapping has recently been used. Fisher fixed point findings in generalised metric spaces were shown by Karim Chaira[17] et al. in 2020 using a graph. In relation to mappings defined on a generalised metric space with a graph, the fisher's fixed point theorem was discussed. The classical Fisher fixed point theorem should be viewed as being extended by this study. It expands on several previous efforts on the generalisation of metric space with graph using the Banach contraction theory. Due to its numerous applications, fixed point theory is now a particularly active topic of study. It relates to the findings that show self-mapping on a set admits a fixed point under specific circumstances. The most well-known result in metric fixed point theory is the Banach Contraction Principle.

- A fixed-point theorem for generalised weakly contractive mappings in b-metric spaces was developed by Eliyas Zinab[18] et al. in 2020. In the context of b-metric spaces, they formulated a fixed-point theorem for generalised weakly contractive mappings and came to the conclusion that a fixed point exists and is unique for self-mappings that meet the theorem.
- Khaled Berrah[19] et al.'s research on the common fixed point in complex valued b-metric space's applications and theorem was published in 2019. For four self-mappings fulfilling rational contraction, they offered a common fixed point theorem, which was shown in complex valued b-metric space. The findings of this study demonstrate that a common solution to the system of Urysohn integral equations and a system of unique solutions to linear equations both exist. Their paper's major goal was to satisfy a rational inequality on complex valued b-metric spaces by presenting common fixed point outcomes of four self-mappings.
- According to the well-known Banach contraction principle (BCP), a contraction mapping over an entire metric space has a single fixed point. Banach used the idea of a diminishing map to arrive at this significant result[20]. The development of computers and new software for rapid and efficient computing has given fixed point theory a new dimension[21]. The Brouwer fixed point theorem is essential for the numerical solution of equations. Exact quotations of the phrase "a continuous map on a close unit ball in Rn has a fixed point" are found in[22].
- Metric spaces' most generic space, which can enable one to reconsider real-world applications, plays a vital part in Real Analysis and Functional Analysis. Understanding and using the idea of topological qualities to normed linear spaces as well as metric space in many domains is always intriguing as well as demanding for mathematicians. Meir-Keeler[23] produced fixed-point solutions using weakly uniformly rigorous contraction in the context of entire metric spaces as a continuation of numerous generalisations. Additionally, there are many other uses for metric fixed point theory, including “dynamic programming, variational inequalities, fractal dynamics, dynamical systems of mathematics, and the placement of satellites in the right orbits for space science”.
- In their research on various fixed point theorems in partial metric spaces with applications, Kanayo Stella Eke and Jimevwo Godwin Oghonyon[24], established a

fixed point theorem for the integral type of these maps. This study claimed that the class of generalised weakly C-contractive mappings in partial metric space and he proved some fixed point results for such maps in ordered partial metric spaces without utilising the continuity of any of the functions. The outcome generalises Chen and Zhu's findings as well as those of other authors in the literature.

- Every function F has at least one fixed point under particular circumstances, as mentioned according to fixed-point theorem. These results have been cited as some of mathematics' most important ones[25]. A fixed point will always be reached by iterating the function in question if the Banach fixed-point theorem is true. Every continuous function on the closed unit ball in n-dimensional Euclidean space has a fixed point according to the Brouwer fixed-point theorem[26]. The theory does not, however, outline how to find the fixed point. The cosine function must have a single, constant value because of its continuation. At a given point, the cosine curve of y cos intersects the plane x, y. The value of this fixed point is 0.739 085 133 215 16[27]. The number of fixed points may be determined using Lefschetz's fixed-point theorem from algebraic topology. In PDE theory, Banach's fixed-point theorem[1] has been generalised[28]. Infinite dimensions can be used to establish fixed point theorems[29]. The collage theorem in fractal compression[30] shows that the fundamental description of a function quickly converges on the desired picture when applied repeatedly to any beginning image.

- A fixed point is a solution to a nonlinear partial differential equation that does not change after the equation has been applied. Therefore, u (x, t) is a fixed point of the operator that defines it if u(x, t) is a solution of a nonlinear PDE[31]. Fixed points play a crucial role in the analysis of nonlinear PDEs because they enable us to comprehend the behaviour of solutions. If we can show that a nonlinear PDE has a unique fixed point[32], it is obvious that only one solution to an equation fulfils particular starting or boundary conditions.

- Fixed point theorems in a novel class of modular metric spaces were the focus of Duran Turkoglu et al.'s[33] research. According to them, they establish a novel idea of generalised modular metric space by taking into account both a modular metric space in the sense of jleli and samet. Then he gives some illustrations to demonstrate that there is a metric structure in the generalised modular metric space. On generalised modular metric spaces, they offered certain fixed-point conclusions for mappings of the contraction and quasi-contraction types.

- Akkouchi[34] established a general common fixed point problem for two pairs {f, S} and {g, T} of weakly compatibles self-maps of a complete b-metric (X, d; s). These maps are satisfying a contractive condition defined by a class of implicit relations in five variables. This contraction unifies, in one go, several contractive conditions previously used in a set of recent papers dealing with fixed point or common fixed results for self-maps of b-metric spaces.
- Harmati IÁ, and Kóczy[35] mentioned that recurrent neural networks called fuzzy cognitive mapping (FCMs) are used to simulate complicated systems utilising weighted causal links. The behaviour of an iteration serves as the basis for inference about the simulated system in FCM-based decision-making. Extensions of fuzzy cognitive maps, fuzzy grey cognitive maps (FGCMs) impose ambiguous weights between the concepts. An iterative process that may converge to an equilibrium point, but may also exhibit limit cycles or chaotic behaviour, determines the inference. In research necessary settings for sigmoid FGCM fixed points' actuality and exceptionality were also very well described.
- Biley[36] discovered a common fixed point theorem for four generalised S-fuzzy metric space self-mappings. The goal of the research was to develop a broader version of the common fixed point theorem that would generalise Singh and Chauhan's finding about the idea of compatibility in fuzzy metric space.
- In order to explain the idea of fuzzy metric space in various forms, Dixit and Gupta[37] developed particular common fixed point theorems for set valued and single valued mappings in fuzzy metric space and fuzzy 2 - metric spaces. The majority of the research's findings either deal with commuting maps or presuppose weak commutativity of mappings. In common fixed point theory, existence theorems were widely proved using the concept of compatible maps. Furthermore, the property E.A in FM-space and the shared fixed points of two incompatible maps were applied. The outcome allowed for the generalisation of several significant fixed point theorems and expanded the field of research for common fixed points under contractive type circumstances.
- Rehman et al.[38] used three-self-mappings to study several coincidence point and common fixed point theorems in fuzzy metric spaces and showed the uniqueness

of some of these conclusions by utilising the weak compatibility of three-self-mappings. The paper also provided some illustrative examples for the validation of the results in support of the findings. The use of fuzzy differential equations to support the work allows for the extension and improvement of several outcomes. Work produced some generalised fuzzy-contraction findings for three weakly compatible self-mappings in FM spaces without making the assumption that the "fuzzy contractive sequences[96] are Cauchy." Further, also demonstrated more coincidence points and CFP outcomes for various contractive type mappings in FM spaces by employing this idea and integral operators.

- Different applications of fixed-point theorems and metric fixed-point theory are made to find a special common solution to differential equations and integral equations. Murthi et al.[39] have demonstrated a few fixed-point theorems in the context of bipolar metric spaces using the extension of Meir-Keeler contraction. Non-trivial examples have been added to the work or the derived findings. In addition to providing an application to discover an analytical solution to an Integral Equation in order to augment the generated result, the results expanded and generalised the conclusions reached in the past.

2.2 COMMON FIXED POINTS APPLICATION FOR COMPATIBLE MAPS

- Several academicians or scholars have used specific connections in the system of fuzzy metric spaces to illustrate standard fixed point theorems. Common fixed-point theorems in generalised fuzzy metric spaces for weakly compatible mappings meeting common E.A. like characteristics. Fixed point assertions in Digital Topology is discussed by Boxer[40, 93, 94] with keeping the freezing sets fits to the theory of the digital topology in order to present the corrections associated with fixed points. The research rephrases almost all valid published statements about digital metric spaces employing the metric rather than the adjacency. Therefore, as a consequence, it appears that the digital metric space is an artificial construct with compromised concern with digital pictures. It was noticed that numerous assertions have been made in the literature on digital metric spaces are counterparts for subsections of Euclidean R^n. The authors frequently overlooked the crucial distinctions between the topological space R^n and digital pictures, leading to false or falsely "proven assertions," trivial, or inconsequential. For instance, many fixed point assertion-satisfying functions essentially be continuous or failed to be continuous digitally.

- Fixed points are strongly connected to these equations since they frequently occur as solutions to nonlinear partial differential equations (PDEs). A fixed point in mathematics is a value that remains constant while the function is applied[41]. A fixed point is a solution that doesn't change after applying the equation while discussing nonlinear partial differential equations. So, if u (x, t) is a nonlinear PDE solution, u (x, t) is a fixed point of the operator that defines it[42]. Fixed points are an important aspect of the study of nonlinear PDEs because they allow us to better understand the behaviour of solutions[43]. If we can show that a nonlinear PDE has a unique fixed point, we know that only one solution to an equation fulfils some initial or boundary conditions[44]. Furthermore, we know that an equation can have more than one solution if we can show that a nonlinear PDE has numerous fixed points. Sometimes, the existence or absence of fixed points can be used to demonstrate the existence or absence of nonlinear PDE solutions. In the analysis of nonlinear PDEs, the fixed point idea is a potent tool that is commonly employed to investigate the existence, uniqueness, and stability of solutions[45].

- Based on the idea of compatible and weakly compatible self-mappings in fuzzy cone metric spaces, research confirmed the use of a few generalised common fixed point theorems for four of these self-mappings[46]. According to research, the conclusions of[47, 48] may be generalised and extended to include self-mappings with continuity of a self-map h and without continuity for a pair of weakly compatible self-mappings.

- Ali et al.'s work[49] employed the ideas of sub-compatibility and sub-sequential continuity to demonstrate a common fixed point theorem for six self-maps in a fuzzy metric space. A number of previous fixed point results in metric space and fuzzy metric space are generalised, expanded, united, and fuzzified by the established result. In the study, a common fixed point theorem for six self-maps in a fuzzy metric space was demonstrated using the ideas of sub-compatibility and sub-sequential continuity. Several fixed point findings in metric space and fuzzy metric space have been generalised, extended, united, and fuzzified. These results may be further extended by expanding the number of self-maps with a new class of inequality. It was investigated if certain full metric spaces of mappings are generically well-posed for fixed point problems, and the very first.

- If and only if x = x is true when f is a mapping from a set or a space X into itself, a point x X is referred to as a fixed point of the mapping[50]. Theorems about fixed points are those that discuss their existence and properties[51]. The most important tools for proving the validity of the solutions to the various mathematical models (differential, integral, partial differential equations, variational inequalities, etc.) that represent various phenomena relevant to various fields, such as steady state temperature distribution, chemical reactions, neutron transport theory, economic theories, epidemics, and fluid flow, are these theorems. They are also used to investigate the problems with optimum control that occur in these systems.

- In 2017, Verma and Shrivastava[52] demonstrated reciprocal continuity for idempotent mappings in fuzzy metric spaces as well as weak commuting in fuzzy metric spaces. This chapter also establishes some conclusions on the existence and uniqueness of fixed point theorems in M-fuzzy metric spaces. Reciprocal continuity for idempotent mappings in M-fuzzy metric spaces, which is shown with examples to supplement our main thesis, has been used to demonstrate the existence of a fixed point theorem.

- The Picard-Banach contractions and some nonexpansive mappings are among the numerous contractive type mappings that belong to the broad class of enriched contractions that Berinde and Pacurar[53] presented. According to research, every enriched contraction has a singular fixed point that may be approximated using the right Krasnoselski iterative approach. The fixed points of local enriched contractions, asymptotic enriched contractions, and enriched contractions of the Maia type were also reported in the research. The research article also included examples to demonstrate the universality of new ideas and accompanying fixed point theorems.

- Under the presumptions that these two pairs of self-maps are weakly compatible and meet a contractive condition, Sekhar[54] demonstrated the presence of shared fixed points between two pairs of self-maps. A series of self-maps is added as an extension of the same. Additionally, the investigation supported the same results with various assumptions on two pairs of self-maps, one of which is weakly compatible and the other of which is compatible and reciprocally continuous. The same results with various hypotheses on two pairs of self-maps, where any of the pair satisfies the (E.A) condition and limits the completeness of X to its subspace, were likewise validated by the study. The research illustrated the extension of Babu and Dula[55] through maps presenting 2-pairs where one pair was weakly companionable. The identical approach is followed for successive self-maps.

- Common fixed point theorems in metric spaces were examined by Semwal and Komal[56] along with its certain scope applications. For specific contractive types of mappings, work examined the existence and uniqueness of common fixed point theorems followed with the advancements. Findings and results are helpful in examining the existence and distinctiveness of common solutions for a set of functional equations that arise in dynamic programming. While studying the various fixed point theorems in partial metric spaces with applications, Eke and Oghonyon[57] for the integral type of these maps established a fixed point theorem. The work claimed class of generalized weakly C-contractive mappings in partial metric space and also proved some fixed point results for such maps in ordered partial metric spaces without utilizing the continuity of any of the functions. The existence and uniqueness of shared fixed points of sometimes weakly compatible mappings meeting particular contractive requirements in a Gsymmetric space were shown by Eke and Oghonyon in 2019[58]. To attain their research goals, they worked upon E. A. Property.

- The proposed primary theorem is a generalised version of a few well-known theorems, as demonstrated by the fact that Patel and Bhardwaj[59] have proven various fixed point and common fixed point theorems for Cone metric space in integral type mappings. The fuzzy metric spaces (Fuzzy 2 and 3 metric spaces) as well as various varieties of fuzzy metric spaces are both amenable to this theorem.

2.3 FUZZY METRIC SPACE, FUZZY MATHEMATICS AND COMMON FIXED POINT

Fuzzy cognitive maps employ directed graphs with edges from the range [1, 1] with constant weights to reflect the direction and intensity of causal linkages. In the FCM theory, the nodes are referred to as "concepts" and represent certain components of the represented system. The numbers in the [0, 1] or [1, 1] interval, known as "activation values," are also used to describe the ideas' present states. In some of the results[60–61] importance of fuzzy logic is addressed for many engineering and related engineering fields, it was indicated that the processing of human perceptions and cognitions served as an inspiration for fuzzy logic, which is founded on the idea of relative graded memberships. Information derived from computational senses and cognitions, which is ambiguous, opaque, imprecise, partially true, or without distinct bounds, can be handled using fuzzy logic. The integration of hazy human judgements in computational problems is made possible by fuzzy logic. For many persons involved in inventive work, such as “engineering, mathematics, computer software, earth science, and physics”, fuzzy logic is incredibly helpful.

- In nonlinear analysis, the metric fixed point theory has been instrumental. It has often entailed the blending of topological and geometrical features. It has been extensively researched and improved upon after the renowned Banach contraction principle, either by altering the contractive condition or the underlying space. By guaranteeing the existence of fixed point, common fixed point, and coincidence point results with various types of applications, such as differential-type applications, integral-type applications, and functional-type applications, many researchers gave generalisation and improved the BCP in many directions for single-valued and multivalued mappings in the context of metric spaces. The notion of fuzzy sets was first proposed by Zadeh[62] in his foundational study, while Goguen[63] subsequently generalised fuzzy sets to L-fuzzy sets. According to the rules of fuzzy logic, certain numbers that are not part of the set are defined as elements within the range [0, 1], in contrast to conventional logic. Zadeh has been able to learn theories of fuzzy sets (FSs) that carry the problem of indefinity thanks to uncertainty, the essential component of genuine difficulty. For a variety of processes, one of which makes use of fuzzy logic, the theory is viewed as a fixed point in the fuzzy metric space (FMS). As a generalisation of fuzzy metric spaces, Park[64] created intuitionistic fuzzy metric spaces. An L-fuzzy fixed point theorem in full metric spaces was proven by Rashid et al.[65]. The fixed points of many fuzzy and L-fuzzy mappings in classical, ordered, fuzzy, and intuitionistic fuzzy metric spaces are then obtained.

- Researchers are constantly interested in learning about new discoveries in metric-space and their potential characteristics. Gahler[66] as a consequence presented the concept of 2-metric spaces, providing the idea of new dimensions for conventional metric spaces. The measure used in this context is non-negative real, or [0, +]; it has several uses. The idea of probabilistic metric spaces, which examines the probabilistic distance between two places, has given the topic and interest in knowing more about stars in the universe a new depth. Similar research was conducted on fuzzy metric spaces by Grabiec[67] and Michalek[68], which considered the degree of agreement and disagreement. The majority of the effort was clearly based on actual figures, whether they are "2-metric, fuzzy metric, modular metric, etc."

- By using fuzzy contractive mappings in non-Archiemedean fuzzy metric space, Mihet[69] established the fixed point theorem and proposed the concept. For two generalisation contractive type mappings, Vetro[70] obtained some Common fixed point solutions. The idea of intuitionistic $(\emptyset, \Psi)$ contractive mappings is explained by Abu-Doniaa et al.[71], along with certain popular fixed point theorems in intuitionistic fuzzy metric space that are proved to be true under these conditions. For compatible and weakly compatible self-mappings obeying the more generalised form of the fuzzy cone Banach contraction theorem in fuzzy cone metric spaces, Rehman et al.[46] found several common fixed point findings. With the condition of Mf triangular, research verified the generalise findings for four self-mappings both with and without a continuous self-map, h.

- The processing of human perceptions and cognitions served as the inspiration for fuzzy logic, which is founded on the idea of relative graded memberships. Information derived from computational senses and cognitions, which is ambiguous, opaque, imprecise, partially true, or without distinct bounds, can be handled using fuzzy logic. The integration of hazy human judgements in computational issues is made possible by fuzzy logic. For many persons involved in inventive work, such as engineering (electrical, chemical, civil, environmental, mechanical, industrial, geological, etc.), mathematics, computer software, earth science, and physics, fuzzy logic is incredibly helpful. Some of their conclusions are presented in[72-74].

- Using implicit relations, Rana et al.[75] developed a few fixed-point theorems for FM-spaces. Through the use of the concepts of compatible maps, implicit relations, weakly compatible maps, and R-weakly compatible maps, several writers have described the number of fixed-point theorems in FMspaces. With the use of continuous t-norms, George and Veeramani[76] refined the idea of FM-spaces and established several fundamental features.

- Sometimes, the existence or absence of fixed points can be used to demonstrate the existence or absence of nonlinear PDE solutions. If we can show that there are no fixed points, then we know that there are no nontrivial solutions to a nonlinear PDE. In the analysis of nonlinear PDEs, the fixed point idea is a potent tool that is widely employed to investigate the existence, uniqueness, and stability of solutions[77]. For mappings meeting the -contractive condition on a fuzzy metric space, several fixed point theorems are proven that are both intuitively stated and compatible with following continuous mappings.

- In fuzzy metric space and fuzzy 2-metric spaces, Dixit and Gupta[37] demonstrated a few basic fixed point theorems for set valued and single valued mappings. The findings of the work dealt either with commuting mappings or assumed the notion of weak commutativity of mappings presented by Seesa. The concept of fuzzy metric space was introduced in many varieties. The outcome of the work allowed for the generalisation of a number of significant fixed point theorems and expanded the field of research for common fixed points under contractive type constraints. In FM-spaces, certain fixed-point theorems have been proven by Rana et al.[78] using implicit relations. In addition, fixed-point theorems have been introduced in FM-spaces utilising the concepts of compatible maps, implicit relations, weakly compatible mappings, and R-weakly compatible maps[41].

- The main goal of this study is to extend the common limit range feature that Gupta et al.[79] presented to V-fuzzy metric spaces. By using this characteristic on V -

fuzzy metric spaces, substantial results for linked maps are also demonstrated. To be more exact, we define the concept of CLR-property for the mappings Θ: $\mathcal{M}\times\mathcal{M}\rightarrow\mathcal{M}$ and Ω: $\mathcal{M}\rightarrow\mathcal{M}$. We present and demonstrate our new fixed point results using our new idea.

- The term "common limit in the range property," or "CLR property," in fuzzy metric spaces was defined by Sintunavarat and Kumam[98]. In contrast to property (E-A), which necessitates this requirement for the existence of the fixed point, the idea of property CLR never needs the closed-ness of the subspace. It is clear that academics are focusing on this property in order to generalise or enhance earlier findings that were supported by the idea of property (E-A). In their publication, Jha and Pant[80] generalised and refined a number of conclusions on fixed point in fuzzy metric space under this condition (E-A) by removing the continuity of mappings even the completeness. Jha and Pant[80] established several common fixed point theorems in fuzzy metric space with property (E-A).
- The exploration of fixed points for contractive mappings is a well-known problem in metric spaces, and Huang and Zhang's presentation of the cone metric space is one such generalisation[81-82]. In addition to substituting an ordered Banach space for the collection of real numbers from a metric space, they provided some crucial results for a self-map satisfying a contractive condition in this space.
- Open spheres and closed spheres are convex in convex metric spaces, and all normed spaces and their convex subsets are convex metric spaces, according to Takahashi[83], who first introduced the concept of convex metric spaces and studied the fixed-point theory for non-expansive mappings in such a setting. Convex metric spaces that are not embedded in a normed space, however, are common. Generalised metric spaces were renamed to G-metric spaces by Mustafa and Sims[84], who also discovered several topological characteristics[95].
- Particularly, Sun and Yang[85] introduced the idea of GFMS and generalised the concept of fuzzy metric spaces. A distinct common fixed point theorem for six weakly compatible mappings in G-fuzzy metric spaces was established by Balasubramanian et al. in 2016[86]. In order to demonstrate how the iteration process converges to the unique fixed point under various contractive mapping conditions on the GFMS in convex structure, a new three-step iteration procedure

is developed in this study effort. The examination into the data dependency on the outcomes of these iterative processes in the generalised G-fuzzy convex metric spaces is also a major emphasis of this chapter. To get fixed point and common fixed point theorems for a pair of self-mappings under suitable contractive type requirements with convex structure, the idea of convex structure in GFMS has also been proposed. Extensions of fuzzy cognitive maps known as fuzzy grey cognitive maps (FGCMs) were developed for the situation when only hazy information was available on the connections between the various components of the system[87]. The iteration may reach a fixed point, enter a limit cycle, or exhibit chaotic patterns, just like the classical FCMs. As a result, the fixed points issue is also very important in the context of FGCMs.

- Pazhani[88] studied "fixed point theorems from fuzzy metric spaces and intuitionistic fuzzy metric spaces" and remarked that one of the most effective and successful nonlinear analysis methods is the Fixed Point Theory, which may be viewed as the nonlinear analysis's core. The adjective "fuzzy" has gained a lot of popularity and usage in recent research pertaining to the logical and set-theoretical underpinnings of mathematics. In a number of application domains, the idea of defining Intuitionistic Fuzzy Sets (IFS) for fuzzy set generalisations has shown to be fascinating and helpful. Research has opened up new possibilities for advancing the theory of intuitionistic fuzzy metric spaces (IFMS) in a path that may be interesting for other branches of mathematics and computer science as well as general topology. With the use of the t-norm, Agnihotri et al.[89] developed the idea of fuzzy metric space. Through the use of weak compatibility, we developed a common fixed point theorem for seven self-mappings in fuzzy metric space. On intuitionistic fuzzy metric space, Abu-Donia et al.[90] established a few often used coupled fixed point theorems for mappings under the ψ-contractive condition under compatible and subsequently continuous mappings.

- A common fixed point theorem for six self-maps in a fuzzy metric space was established by Ali et al.[91] combining the ideas of sub-compatibility and sub-sequential continuity. A number of previous fixed point results in metric space and fuzzy metric space are generalised, expanded, united, and fuzzified by the established result.

- In order to realize the weak compatible mappings (wc-mapping), Vijayalakshmi et al.[92] demonstrated a fixed-point technique on E -Fuzzy-metric Space, offering Joint Common Limit in the Range (JCLR)-property implicitly. Key conclusions from the research were shown with several specific cases. For six finite families of self-mappings, work demonstrated a fixed-point theorem that may be used to support additional conclusions. Furthermore, it was demonstrated that typical fixed-point theorems may be proved using any finite number of mappings[97]. Analysis and applied mathematics both heavily rely on metrics that are typically thought of as functions of distance. But in other situations, like as the determination of the distance between two pixels in image analysis, which is frequently regarded as two-pixel similarity, metrics based on the Crisp notion are not appropriate. Therefore, based on fuzzy notions, Lukman Zicky et al.[99] established the idea of fuzzy metric. Then, convergence and fixed point issues were addressed using this fuzzy metric. Some characteristics of regular metric still hold true for fuzzy metric thanks to work.

- Hasan[100] made the most of the variable distance function by identifying some general fixed point theorems for tangential mappings for hybrid couples of both sorts of mappings (single and multi-valued). A number of earlier recognised discoveries were expanded upon and generalised by these theorems. The amount of conclusion generalisation and the veracity of assumptions were both checked by the author.

❐❐❐

REFERENCES

[1.] S. Banach, Sur les operations dans les ensembles abstraits et leur appliacation aux equations integrales, Fundam. Math. 3(1922), 133-181 (in French).

[2.] L. Boxer and P.C. Staecker, Remarks on fixed point assertions in digital topology, Applied General Topology 20 (1) (2019); 135-153. Online available at: https://polipapers.upv.es/ index.php/AGT/article/view/10474/11201

[3.] S. Mukherjee, N. Kumar, A. A. Bezrukov, K. Tan, T. Pham, K.A. Forrest, K.A. Oyekan, O.T. Qazvini, D.G. Madden, B. Space, M.J. Zaworotko, Angew. Chem. 133 (19), 2021; 10997–11004, doi: 10.1002/ange.202100240.

[4.] M. F. Alotaibi, M. Omri, E. Khalil, S. Abdel-Khalek, J. Bouslimi, M.M. Khater, J. Ocean Eng. Sci., 2022.

[5.] M.M. Khater, J. Ocean Eng. Sci.,2022.

[6.] F. Wang, S. Muhammad, A. Al-Ghamdi, M. Higazy, M.M. Khater, J. Ocean Eng. Sci.,2022.

[7.] Y. Jiang, F. Wang, M.M. Khater, A.G. Al-Sehemi, S. Ullah, O.A. Al-Hartomy, M. Higazy, J. Ocean Eng. Sci., 2022.

[8.] J. Zhang, D. Lu, S.A. Salama, M.M. Khater, AIP Adv 12 (2), 2022; 025015.

[9.] M. Omri, A.-H. Abdel-Aty, S. Abdel-Khalek, E. Khalil, M.M. Khater, Alex. Eng. J. 61 (9), 2022; 6 887–6 895.

[10.] F. Wang, S. Muhammad, A. Al-Ghamdi, M. Higazy, M.M. Khater, J. Ocean Eng. Sci.,2022.

[11.] H. Ma, F. Wang, M.M. Khater, A.G. Al-Sehemi, M. Pannipara, O.A. Al-Hartomy, M. Higazy, J. Ocean Eng. Sci., 2022.

[12.] M.M. Khater, R.A. Attia, A.-H. Abdel-Aty, M. Abdou, H. Eleuch, D. Lu, Results Phys. 16, 2020; 1030 0 0.

[13.] A.T. Ali, M.M. Khater, R.A. Attia, A.-H. Abdel-Aty, D. Lu, Chaos Solitons Fractals 131, 2020; 109473.

[14.] C. Park, M.M. Khater, A.-H. Abdel-Aty, R.A. Attia, H. Rezazadeh, A. Zidan, A.-B. Mohamed, Alex. Eng. J. 59 (3), 2020; 1425–1433.

[15.] L. Chen, Li C, R. Kaczmarek, Y. Zhao. Several fixed point theorems in convex b-metric spaces and applications. Mathematics. 2020 Feb 14; 8(2):242.

[16.] B. Fisher, Common Fixed point mappings, Ind. J. Math pure. Appl. Math, 20, 1978; 2-16.

[17.] K. Chaira, A. Eladraoui, M. Kabil, A. Kamouss. Fisher fixed point results in generalized metric spaces with a graph. International journal of mathematics and mathematical sciences. 2020 Mar 16; 2020:1-8.

[18.] E. Zinab, K. Koyas, A. Girma. A fixed point theorem for generalized weakly contractive mappings in b-metric spaces. Open journal of mathematical sciences. 2020 Dec 31; 4(1):1-8.

[19.] K. Berrah, A. Aliouche, T. Oussaeif. Applications and theorem on common fixed point in complex valued b-metric space. AIMS mathematics. 2019 Jan 1; 4(3):1019-33.

[20.] J. Shalf. The future of computing beyond Moore's Law. Philosophical Transactions of the Royal Society A. 2020 Mar 6; 378(2166):20190061.

[21.] S. Khalehoghli, H. Rahimi, Gordji ME. Fixed point theorems in R-metric spaces with applications. AIMS Mathematics. 2020 Mar 10;5(4):3125-37.

[22.] M. Shulman. Brouwer's fixed-point theorem in real-cohesive homotopy type theory. Mathematical Structures in Computer Science. 2018 Jun; 28(6):856-941.

[23.] A. Meir; E. Keeler. A theorem on contraction mappings. J. Math. Anal. Appl. 1969, 28, 326–329.

[24.] KS. Eke, JG. Oghonyon. Some fixed point theorems in ordered partial metric spaces with applications. Cogent Mathematics & Statistics. 2018 Jan 1; 5(1):1509426.

[25.] X. Wu, T. Wang, P. Liu, G. Deniz Cayli, and X. Zhang, Topological and algebraic structures of the space of Atanassov's intuitionistic fuzzy values, arXiv:2111.12677, 2021.

[26.] J.-W. Cao, S. Mukherjee, T. Pham, Y. Wang, T. Wang, T. Zhang, X. Jiang, H.-J. Tang, K. A. Forrest, B. Space, M. J. Zaworotko, and K.-J. Chen, One-step ethylene production from a four-component gas mixture by a single physisorbent, Nat. Commun. 12, 6507,2021.

[27.] Y. L. Peng, T. Wang, C. Jin, C.-H. Deng, Y. Zhao, W. Liu, K. A. Forrest, R. Krishna, Y. Chen, T. Pham, B. Space, P. Cheng, M. J. Zaworotko, and Z. Zhang, Efficient propyne/propadiene separation by microporous crystalline physiadsorbents, Nat. Commun. 12, 5768, 2021.

[28.] M. Shivanna, K. i. Otake, B. Q. Song, L. M. Wyk, Q. Y. Yang, N. Kumar, W. K. Feldmann, T. Pham, S. Suepaul, B. Space, L. J. Barbour, S. Kitagawa, and M. J. Zaworotko, Benchmark acetylene binding affinity and separation through induced fit in a flexible hybrid ultramicroporous material," Angew. Chem. 60(37), 20383–20390, 2021.

[29.] S. Mukherjee, N. Kumar, A. A. Bezrukov, K. Tan, T. Pham, K. A. Forrest, K. A. Oyekan, O. T. Qazvini, D. G. Madden, B. Space, and M. J. Zaworotko, "Amino functionalised hybrid ultra-microporous materials that enable single-step ethylene purification from a ternary mixture, Angew. Chem. 133(19), 10997–11004, 2021.

[30.] Z. Niu, X. Cui, T. Pham, G. Verma, P. C. Lan, C. Shan, H. Xing, K. A. Forrest, S. Suepaul, B. Space, A. Nafady, A. M. Al-Enizi, and S. Ma, Frontispice: A MOFbased ultra-strong acetylene nano-trap for highly efficient C2H2/CO2 separation, Angew. Chem. 133(10), 202181062, 2021.

[31.] M. M. A. Khater, Nonlinear biological population model; computational and numerical investigations," Chaos, Solitons Fractals 162, 112388, 2022.

[32.] M. M. A. Khater, Multi-vector with nonlocal and non-singular kernel ultrashort optical solitons pulses waves in birefringent fibers, Chaos, Solitons Fractals 167, 113098, 2023.

[33.] D. Turkoglu, N. Manav. Fixed point theorems in a new type of modular metric spaces. Fixed Point Theory and Applications. 2018 Dec; 2018: 1-0.

[34.] M. Akkouchi, A common fixed point result for two pairs of maps in b-metric spaces without (E.A.)-property. Mathematica Moravica. 23. 2019; 29-44. 10.5937/MatMor1902029A.

[35.] IÁ Harmati, and Kóczy LT. On the convergence of sigmoidal fuzzy grey cognitive maps. International Journal of Applied Mathematics and Computer Science. 2019; 29(3):453-66.

[36.] S. Biley. Common fixed point theorems for four mappings in fuzzy metric space. International Journal of Multidisciplinary Trends 2022; 4(1): 92-96.

[37.] S. Dixit and Gupta AK. Common Fixed Point Theorem in ϵ-Chainable Fuzzy Metric Space Using Absorbing Maps. Scandinavian Journal of Information Systems. 2023; 35 (3):117-124.

[38.] SU. Rehman, I. Shamas, N. Jan, A. Gumaei, Al-Rakhami M. Some coincidence and common fixed point results in fuzzy metric space with an application to differential equations. Journal of Function Spaces. 2021 Oct 29; 2021:1-5.

[39.] P.P. Murthy, C.P. Dhuri, S. Kumar, R. Ramaswamy, Alaskar, M.A.S., Radenovi'c, S. Common Fixed Point for Meir–Keeler Type Contraction in Bipolar Metric Space. Fractal Fract. 2022, 6, 649. https://doi.org/10.3390/fractalfract6110649.

[40.] L. Boxer. Remarks on Fixed Point Assertions in Digital Topology, 7. arXiv preprint arXiv:2307.04969, 2023 Jul 11.

[41.] M. M. A. Khater, X. Zhang, and R. A. M. Attia, "Accurate computational simulations of perturbed Chen–Lee–Liu equation," Results Phys. 45, 106227, 2023.

[42.] M. M. A. Khater, Computational and numerical wave solutions of the Caudrey–Dodd–Gibbon equation, Heliyon 9(2), e13511, 2023.

[43.] M. M. A. Khater, S. H. Alfalqi, J. F. Alzaidi, and R. A. M. Attia, Novel soliton wave solutions of a special model of the nonlinear Schrödinger equations with mixed derivatives, Results Phys. 47, 106367, 2023.

[44.] M. M. A. Khater, "Multi-vector with nonlocal and non-singular kernel ultrashort optical solitons pulses waves in birefringent fibers," Chaos, Solitons Fractals 167, 113098, 2023.

[45.] M. M. A. Khater, "Physics of crystal lattices and plasma; analytical and numerical simulations of the Gilson–Pickering equation," Results Phys. 44, 106193, 2023.

[46.] SU. Rehman, S. Jabeen, F. Abbas, H. Ullah, I. Khan. Common fixed point theorems for compatible and weakly compatible maps in fuzzy cone metric spaces. Annals of Fuzzy Mathematics and Informatics. 2020 Feb; 19(1):1-9.

[47.] S. U. Rehman and H. X. Li, Fixed point theorems in fuzzy cone metric spaces, J. Nonlinear Sci. Appl., 10, 2017; 5763–5769.

[48.] S. U. Rehman, Y. Li, S. Jabeen and T. Mahmood, Common fixed point theorems for a pair of self-mappings in fuzzy cone metric spaces, Abstract and Applied analysis, 2019, 10 pages.

[49.] S. S. Ali, N. Kumari, R. Kumar, An Extended Result on Sub-compatible and Sub-Sequential Continuous Maps in Fuzzy Metric Space, 20(1), 2022; 17-26.

[50.] S. Sedghi, A. Gholidahneh & K. P. Rao. Common fixed point of two R-weakly commuting mappings in Sb-metric spaces. Math. Sci. Lett, 6 (3), 2017; 249–253.

[51.] A. A. N. Abdou. Common fixed point results for compatible-type mappings in multiplicative metric spaces. J. Nonlinear Sci. Appl., 9, 2016; 2244–2257.

[52.] N. Verma and R. Shrivastava. Fixed point theorem in fuzzy metric space for idempotent mappings, International Journal of theoretical & Applied Sciences, 9(2), 2017; 201-204.

[53.] V. Berinde, M. Păcurar. Approximating fixed points of enriched contractions in Banach spaces. J. Fixed Point Theory Appl. 22, 38, 2020. https://doi.org/10.1007/s11784-020-0769-9.

[54.] S. Bhanu. Common fixed points of generalized contraction maps in metric spaces, J. Math. Comput. Sci., 9, 2019; 254-287.

[55.] G.V.R. Babu and T.M. Dula. Fixed points of generalized contraction maps in metric spaces, Adv. Fixed Point Theory, 7(1), 2017; 97-117.

[56.] P. Semwal and Komal, Common fixed point theorems in metric spaces with applications". European journal of pure and applied mathematics. 11.4:1177-1190; 2018.

[57.] K. S. Eke and J. G. Oghonyon, Some fixed point theorems in ordered partial metric spaces with applications". Cogent mathematics and statistics. 5: 2574-2558; 2018.

[58.] K. S. Eke and J. G. Oghonyon, Common Fixed Point Results for Generalized

Contractive Maps in G-Symmetric Spaces, International Journal of Mechanical Engineering and Technology, 10 (02), 2019, 1688–1698.

[59.] S. T. Patel, R. Bhardwaj. Some Common Fixed Point Theorem For Cone Metric Space In Integral Type, International Journal of Research In Modern Engineering And Emerging Technology, 1(1), January 2013.

[60.] P. O. Mohammed, H. Aydi, A. Kashuri, Y. S. Hamed, and K. M. Abualnaja, "Midpoint inequalities in fractional calculus defined using positive weighted symmetry function kernels," Symmetry, vol. 13, no. 4, p. 550, 2021.

[61.] K. Javed, F. Uddin, H. Aydi, A. Mukheimer, and M. Arshad, "Ordered-theoretic fixed point results in fuzzy b-metric spaces with an application," Journal of Mathematics, vol. 2021, Article ID 6663707, 7 pages, 2021.

[62.] L. A. Zadeh, Fuzzy set, Information and Control, 8, 1965; 338–353. 1

[63.] J. A. Goguen, L-fuzzy sets, J. Math. Anal. Appl., 18, 1967; 145–174. 1, 2.12

[64.] J. H. Park, Intuitionistic fuzzy metric spaces, Chaos Solitons Fractals, 22, 2004; 1039–1046. 1

[65.] M. Rashid, A. Azam, N. Mehmood, L-Fuzzy Fixed Points Theorems for L-Fuzzy Mappings via B - =L-Admissible Pair, Sci. World J., 2014; 8 pages. 1, 2.13, 2.14

[66.] S. Gahler. 2-metricsche Raume und ihre topologische struktur. Math. Nachr. 1963; 26, 115–148.

[67.] M. Grabiec. Fixed points in fuzzy metric space. Fuzzy Sets Syst. 1988; 27, 385–389.

[68.] I. Kramosil; J. Michalek. Fuzzy metric and statistical metric spaces. Kybernetica 1975; 15, 326–334.

[69.] D. Mihet¸, Fuzzy–contractive mappings in non–Archimedean fuzzy metric spaces, Fuzzy Sets and Systems, 159, 2008; 739–744. 1

[70.] C. Vetro, Fixed points in weak non-Archimedean fuzzy metric spaces, Fuzzy Sets and Systems, 162, 2011; 84–90. 1

[71.] H. M. Abu-Doniaa, H. A. Atiaa, M. A. Khaterb Omnia. Common fixed point theorems in intuitionistic fuzzy metric spaces and intuitionistic $(\emptyset, \Psi)$-

contractive mappings, J. Nonlinear Sci. Appl., 13, 2020; 323-329.

[72.] A. Bajpai and V. S. Kushwah, Importance of fuzzy logic and application areas in engineering research, International Journal of Recent Technology and Engineering, 7, 1467–1471, 2019.

[73.] P. O. Mohammed, H. Aydi, A. Kashuri, Y. S. Hamed, and K. M. Abualnaja, Midpoint inequalities in fractional calculus defined using positive weighted symmetry function kernels, Symmetry, 13 (4), 550, 2021.

[74.] K. Javed, F. Uddin, H. Aydi, A. Mukheimer, and M. Arshad, Ordered-theoretic fixed point results in fuzzy b-metric spaces with an application, Journal of Mathematics, 2021, Article ID 6663707, 7 pages, 2021.

[75.] R. Rana, R. C. Dimri, and A. Tomar, "Fixed point theorems in fuzzy metric spaces using implicit relations," International Journal of Computer Applications, 8 (1), 16–21, 2010.

[76.] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64 (3), 395–399, 1994.

[77.] M. M. A. Khater, Hybrid accurate simulations for constructing some novel analytical and numerical solutions of 3-order GNLS equation, Int. J. Geom. Methods Mod. Phys. (published online, 2023).

[78.] R. Rana, R. C. Dimri, and A. Tomar, Fixed point theorems in fuzzy metric spaces using implicit relations, International Journal of Computer Applications, vol. 8, no. 1, pp. 16–21, 2010.

[79.] V. Gupta, W. Shatanawi, A. Kanwar. Coupled Fixed Point Theorems Employing CLR-Property on V -Fuzzy Metric Spaces. Mathematics 2020; 8:404. https://doi.org/10.3390/math8030404.

[80.] K. Jha and V. Pant, Some common fixed point theorem in fuzzy metric space with property (E-A), Thai J. Math. 15 (1), 2017; 217-226.

[81.] R. Verma, Q. A. Kabir, M. S. Chauhan, & R. Shrivastava. Generalized Fixed point Theorem for multi-valued contractive mapping in cone b-metric space. Advances and Applications in Mathematical Sciences, 19 (8), 2020; 825–832.

[82.] Ali Abou Bakr, S. M., Theta cone metric spaces and some fixed point theorems. Journal of Mathematics, 2020; 1–7.

[83.] W. Takahashi. A convexity in metric space and non-expansive mappings, I Kodai Math. Sem. Rep. 22, 1970; 142-149.

[84.] Z. Mustafa and B. Sims. A new approach to generalized metric space, Journal of Nonlinear and Convex Analysis, vol 7, 2006; no. 2, 289-297.

[85.] G. Sun and K. Yang. Generalized fuzzy metric spaces with properties, Res. J. App. Sci. Engg. and Tech., Vol.2, 2010; 673-678.

[86.] G. Balasubramanian, M. Rajeswari, M. Jeyaraman. Common fixed point theorems in intuitionistic generalized fuzzy metric spaces, Journal of Advanced studies in topology, 2016; 68-78.

[87.] L. Jose Salmeron. Modelling grey uncertainty with Fuzzy Grey Cognitive Maps, Expert Systems with Applications, 37, 12, 2010; 7581-7588, https://doi.org/10.1016/j.eswa. 2010.04.085.

[88.] V. Pazhani, Studies On Fixed Point Theorems from Fuzzy Metric Spaces and Intuitionistic Fuzzy Metric Spaces, PG and Research Department of Mathematics K. N. Govt. Arts College for Women (Autonomous)Thesis, 2021.

[89.] S. Agnihotri, KK. Dubey, VK Gupta. Common Fixed Point Theorems in Fuzzy Metric Space. International Journal of Applied Engineering Research. 2019;14(11):2738-47.

[90.] H.M. Abu-Donia, M.S. Bakry, H.A. Atia et al., Studying triple fixed point through the modified intuitionistic fuzzy metric space, Journal of Ocean Engineering and Science, 2022, 1-7. https://doi.org/10.1016/j.joes.2022.05.030.

[91.] SS Ali, N. Kumari, R. Kumar. An Extended Result on Sub-compatible and Sub-Sequential Continuous Maps in Fuzzy Metric Space. Intern. J. Fuzzy Mathematical Archive, 20 (1); 2022; 17-26.

[92.] C. Vijayalakshmi, P. Muruganantham, AN. Gani. Fixed-Point Approach on Ě Fuzzy-Metric Space with JCLR Property by Implication. InInternational Conference on Educational Technology and Administration 2022 Dec 16 (pp.

63-69). Singapore: Springer Nature Singapore.

[93.] L. Boxer, Remarks on Fixed Point Assertions in Digital Topology, 2, Applied General Topology 20, (1), 2019; 155-175. Online available at: https://polipapers.upv.es/index.php /AGT/article/view/10667/11202

[94.] L. Boxer, Remarks on Fixed Point Assertions in Digital Topology, 3, Applied General Topology 20 (2), 2019; 349-361. Online available at: https://polipapers.upv.es/index.php /AGT/article/view/11117

[95.] X. Wu, T. Wang, P. Liu, G. Deniz Cayli, X. Zhang, Topological and algebraic structures of the space of Atanassov's intuitionistic fuzzy values, arXiv:2111. 12677, 2021.

[96.] P.K. Sharma, Some Common Fixed Point Theorems for Sequence of Self Mappings in Fuzzy Metric Space with Property (CLRg), J. Math. Comput. Sci. 10, 2020; No. 5, 1499-1509, https://doi.org/10.28919/jmcs/4650.

[97.] S. Reich, A.J. Zaslavski. Generic Well-Posedness of Fixed Point Problems. Vietnam J. Math. 46, 5–13, 2018. https://doi.org/10.1007/s10013-017-0251-1.

[98.] W. Sintunavarat and P. Kumam, Common Fixed Point Theorems for a Pair of Weakly Compatible Mappings in Fuzzy Metric Spaces, Journal of Applied Mathematics, 2011. https://doi.org/10.1155/2011/637958

[99.] L. Zicky, IG Usadha, Sunarsini. Fixed Point Theorem On Fuzzy Metric Space. In Journal of Physics: Conference Series, 2019, 1218 (1), 012062. IOP Publishing.

[100.] M. Hasan. Common Fixed Point Theorems for Hybrid Pairs of Maps in Fuzzy Metric-Like Spaces by Distance Adjustment, Eur. Chem. Bull. 2023; 12 (1), 1786 – 1794. DOI: 10.31838/ecb/2023.12.1.260.

CHAPTER – III

FUZZY METRIC SPACE

CONTENTS

3.1 INTRODUCTION AND PRELIMINARIES OF COMPLETE MULTIPLICATIVE METRIC SPACE

Since its introduction by Banach in 1922[1], the Banach contraction principle has sparked considerable interest in the study of fixed and common fixed point theorems for maps[2]. Over the years, scholars have extended this principle to various spaces, such as quasi-metric, fuzzy metric, 2-metric, cone metric, partial metric, and generalized metric spaces[3]. In 2008, Bashirov proposed the concept of multiplicative metric spaces and delved into multiplicative calculus, culminating in the establishment of its fundamental theorem. Building upon this foundation, in 2012, Florack and Assen explored the application of multiplicative calculus in the analysis of biological images[4].

Definition 3.1:[5] Consider a nonempty set A. A multiplicative metric is a function d:$A\times A\rightarrow$R+ that fulfills the following conditions:

1. $d(a,b)\geq 1$ for all $a,b\in A$, with equality $d(a,b)=1$ if and only if $a=b$ (referred to as ($M1$).
2. $d(a,b)=d(b,a)$ for all $a,b\in A$ (denoted as ($M2$)).
3. $d(a,b)\leq d(a,c)\cdot d(c,b)$ for all $a,b,c\in A$ (satisfying the multiplicative triangle inequality) ($M3$). The pair (A,d) forms a multiplicative metric space.

Definition 3.2:[6] Given a multiplicative metric space (A,d), a sequence $\{a_n\}$ in A, and $a\in A$, if for every multiplicative open ball $B(a)=\{b|d(a,b)<\epsilon\}$ with $\epsilon>1$, there exists a natural number $N\in$N such that $n\geq N$ implies $a_n\in B(a)$, then $\{a_n\}$ is termed multiplicative convergent to a, denoted as $a_n\rightarrow a$ as $n\rightarrow\infty$.

Proposition 3.1:[7] For a multiplicative metric space (A,d), a sequence $\{a_n\}$ in A, and $a\in A$, $a_n\rightarrow a$ as $n\rightarrow\infty$ if and only if $d(a_n,a)\rightarrow 1$ as $n\rightarrow\infty$.

Definition 3.3: Let (A,d) be a multiplicative metric space[8] and $\{a_n\}$ be a sequence in A. The sequence $\{a_n\}$ is labelled a multiplicative Cauchy sequence if, for every $\epsilon>1$, there exists a positive integer $N\in$N such that $d\ (a_n,\ a_m)<\epsilon$ for all $n,m\geq N$.

Proposition 3.2: For a multiplicative metric space[8] (A,d) and a sequence $\{a_n\}$ in A, $\{a_n\}$ is a multiplicative Cauchy sequence if and only if $d(a_n,a_m)\rightarrow 1$ as $n,m\rightarrow\infty$.

Definition 3.4: A multiplicative metric space[8] (A,d) is declared multiplicative complete if every multiplicative Cauchy sequence in (A,d) is multiplicative convergent in A.

Definition 3.5: Let (A,d_A) and (B,d_B) be two multiplicative metric spaces[9], and $f{:}A\to B$ be a function. f is termed multiplicative continuous at $a\in A$ if for every $\epsilon>1$, there exists $\delta>1$ such that $f(B_\delta(a))\subset B_\delta(f(a))$.

Proposition 3.3: For multiplicative metric spaces[9] (A,d_A) and (B,d_B), a mapping $f{:}A\to B$, and any sequence $\{a_n\}$ in A, f is multiplicative continuous at $a\in A$ if and only if $f(a_n)\to f(a)$ for every sequence $\{a_n\}$ with $a_n\to a$ as $n\to\infty$.

Proposition 3.4: Given a multiplicative metric space (A,d_A), sequences $\{a_n\}$ and $\{b_n\}$ in A such that $a_n\to a$ and $b_n\to b$ as $n\to\infty$, where $a,b\in A$, $d(a_n,b_n)\to d(a,b)$ as $n\to\infty$.

Definition 3.6: The self-maps f and q of a set A are called commutative if $fqa=qfa$ for all $a\in A$.

Definition 3.7: Suppose f and q are two self-mappings of a multiplicative metric space (A,d)[10]. The pair (f,q) are called weak commutative mappings if $d(fqa,qfa)\leq d(fa,qa)$ for all $a\in A$.

Definition 3.8: Let (A,d) be a multiplicative metric space, and let $f{:}A\to A$ be called a multiplicative contraction if there exists a real constant $\lambda\in(0,1)$ such that $d(f(a),f(b))\leq d(a,b)\lambda$ for all $a,b\in A$.

Theorem 3.1: Let (A,d) be a multiplicative metric space, and let $f{:}A\to A$ be a multiplicative contraction. If (A,d) is complete, then f has a unique fixed point.

Theorem 3.2: Let P,Q,M, and N be self-mappings of a multiplicative metric space A, they satisfy the following conditions:

- $P(A)\subset N(A)$, $Q(A)\subset M(A)$;
- M and P are weak commutative, N and Q are also weak commutative;
- one of P, Q, M, and N is continuous;
- $d(Pa,Qb)\leq\{\max\{d(Ma,Nb),d(Ma,Pa),d(Nb,Qb),d(Pa,Nb),d(Ma,Qb)\}\}$, $\lambda\in(0,1/2)$, for all $a,b\in A$. Then P, Q, M, and N have a unique common fixed point.

Definition 3.9: The self-maps f and q of a multiplicative metric space[11] (A,d) are said to be compatible if $\lim_{m\to\infty} d(fq(a_m),qf(a_m))=1$, whenever $\{a_m\}$ is a sequence in A such that $\lim_{m\to\infty} fa_m=\lim_{m\to\infty} qa_m=t$, for some $t\in A$.

Definition 3.10: Suppose that f and q are two self-maps of a multiplicative metric space (A, d). The pair (f, q) are called weakly compatible mappings if $f(a)=q(a)$ for $a\in A$ implies $fq(a)=qf(a)$. That is, $d(fa,qa)=1$ implies $d(fq(a),qf(a))=1$.

Remark 3.1: Commutative mappings must be weak commutative mappings, weak commutative mappings must be compatible, compatible mappings must be weakly compatible, but the converse is not true.

Example 3.1: Let A=R and (A,d) be a multiplicative metric space defined by $d(a,b)=e^{|a-b|}$ for all a,b in A. Let f and q be two self-mappings defined by $f(a)=a^3$ and $q(a)=2-a$. Then $d(f(a_m),q(a_m))= e^{|a_m-1|.|a_{2m}+a_m+2|} \to 1$ if $a_m\to 1$.

iff $a_m \to$d $(fqa_m, qfa_m)=e^{6|a_m-1|^2} = 1$ if $a_m\to 1$

Thus f and q are compatible. Note that $d(fq(0),qf(0))=d(8,2)=e^6>e^2=d(0,0)=d(f(0),q(0))$, so the pair (f,q) is not weakly commuting.

Example 3.2: Let $A=[0,+\infty)$, (A,d) be a multiplicative metric space defined by $d(a,b)=e^{|a-b|}$ for all a,b in A. Let f and q be two self-mappings defined by:

$\mathrm{fa} = \{a, \text{if } 0 \le a < 2, 2, \text{if } a = 24, \text{if } 2 < a < +\infty$

$\mathrm{ga} = \{4 - a, \text{if } 0 \le a < 2, 2, \text{if } a = 27, \text{if } 2 < a < +\infty$

By the definition of the mappings of f and q, only for $a=2$, $fa=qa=2$, at this time $fqa=qfa=2$, so we see the pair (f,q) is weakly compatible. For $a_m=2-1/m\in(0,2)$, from the definition of the mappings of f and q we have $f(a_m)=q(a_m)=2$, but $d(fq(a_m),qf(a_m))=e^{a_m}=e^2\neq 1$, so the pair (f,q) is not compatible.

Let $\emptyset$ denote the set of functions $\varphi:[1,\infty]^5\to[0,\infty)$ satisfying:

- φ is non-decreasing and continuous in each coordinate variable;
- for ≥ 1,for $t\geq 1$, $\psi(t)=\max\{\varphi(t,t,t,1,t),\varphi(t,t,t,t,1),\varphi(t,1,1,t,t),\varphi(1,t,1,t,1),\varphi(1,1,t,1,t)\}\leq t$. From now on, unless otherwise stated, we choose $\varphi \in \emptyset$.

THEOREM 3.2: *Let* (A, d) *be a complete multiplicative metric space*, P, Q, M, *and* N *be four mappings of* A *into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that* $P(X) \subset N(X)$, $Q(X) \subset M(X)$, *and*

$$d(Pa, Qb) \leq \varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb) \quad ..(3.1)$$

for all $a, b \in A$. *Assume one of the following conditions is satisfied*:

a. *Either* M *or* P *is continuous*, *the pair* (P, M) *is compatible and the pair* (Q, N) *is weakly compatible*.

b. *Either* N *or* Q *is continuous*, *the pair* (Q, N) *is compatible and the pair* (P, M) *is weakly compatible*.

Then P, Q, M, *and* N *have a unique common fixed point*.

Proof : Let $a_0 \in A$. Since $P(A) \subset N(A)$ and $Q(A) \subset M(A)$, there exist $a_1, a_2 \in A$ such that $y_0 = Pa_0 = Na_0$ and $y_1 = Qa_1 = Ma_1$. By induction, there exist sequences $\{a_n\}$ and $\{y_n\}$ in A such that

$$Y_{2n} = Pa_{2n} = Na_{2n+1}, y_{2n+1} = Qa_{2n+1} = Ma_{2n+2} \quad ..(3.2)$$

for all $n = 0, 1, 2,\dots$.

Next, we prove that $\{y_n\}$ is a multiplicative Cauchy sequence in A. In fact, $\forall n \in$ N, from (3.1), (3.2), and the property of ψ we have

$$d(y_{2n}, y_{2n+1}) = d(Pa_{2n}, Qa_{2n+1})$$

$$\leq \varphi\, d^{\lambda}(Ma_{2n}, Na_{2n+1}), d^{\lambda}(Ma_{2n}, Pa_{2n}), d^{\lambda}(Na_{2n+1}, Qa_{2n+1}),$$

$$d^{\lambda}(Pa_{2n}, Na_{2n+1}), d^{\lambda}(Ma_{2n}, Qa_{2n+1})$$

$$= \varphi\, d^{\lambda}(y_{2n-1}, y_{2n}), d^{\lambda}(y_{2n-1}, y_{2n}), d^{\lambda}(y_{2n}, y_{2n+1}), d^{\lambda}(y_{2n}, y_{2n}), d^{\lambda}(y_{2n-2}, y_{2n+1})$$

$$\leq \varphi\, d^{\lambda}(y_{2n-1}, y_{2n}), d^{\lambda}(y_{2n-1}, y_{2n}), d^{\lambda}(y_{2n}, y_{2n+1}), 1,$$

$$d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1})$$

$$\leq \varphi\, d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1}), d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1}),$$

$$d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1}), 1, d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1})$$

$\leq \psi\ d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1})$

$\leq d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1})$.

This implies that

$$d(y_{2n}, y_{2n+1}) \leq d^{\lambda/1-\lambda}(y_{2n-1}, y_{2n}) = d^{h}(y_{2n-1}, y_{2n}). \qquad ..(3.3)$$

$h = \lambda/1- \lambda \in (0, 1)$.

Similarly, using (3.1), (3.2), and the property of ψ, we have

$d(y_{2n+1}, y_{2n+1}) = d(Qa_{2n+1}, Pa_{2n+1}) = d(Pa_{2n+1}, Qa_{2n+1})$

$\leq \varphi\ d^{\lambda}(Ma_{2n+1}, Na_{2n+1}), d^{\lambda}(Ma_{2n+1}, Pa_{2n+1}), d^{\lambda}(Na_{2n+1}, Qa_{2n+1}), d^{\lambda}(Pa_{2n+1}, Na_{2n+1}), d^{\lambda}(Ma_{2n+2}, Qa_{2n+1})$

$= \varphi\ d^{\lambda}(y_{2n+2}, y_{2n}), d^{\lambda}(y_{2n+1}, y_{2n+1}), d^{\lambda}(y_{2n}, y_{2n+1}), d^{\lambda}(y_{2n+2}, y_{2n}), d^{\lambda}(y_{2n+1}, y_{2n+1})$

$\leq \varphi\ d^{\lambda}(y_{2n}, y_{2n+1}), d^{\lambda}(y_{2n+1}, y_{2n+1}), d^{\lambda}(y_{2n}, y_{2n+2}), d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+1}, y_{2n+2}), 1$

$\leq \varphi\ d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+1}, y_{2n+2}), d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+2}, y_{2n+1}), d^{\lambda}(y_{2n}, y_{2n+2}) \cdot d^{\lambda}(y_{2n+2}, y_{2n+1}), d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+1}, y_{2n+2}), 1)$

$\leq \psi\ d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+1}, y_{2n+2})$

$\leq d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+1}, y_{2n+2})$.

This implies that

$$d(y_{2n+1}, y_{2n+2}) \leq d^{\lambda/1-\lambda}(y_{2n}, y_{2n+1}) = d\ (y_{2n}, y_{2n+1}) \qquad ..(3.4)$$

It follows from (3.3) and (3.4) that, for all $n \in$ N, we have

$$d(y_n, y_{n+1}) \leq d^{h}(y_{n-1}, y_n) \leq d^{h_2}(y_{n-2}, y_{n-1}) \leq \cdot\cdot\cdot \leq d^{h_n}(y_0, y_1).$$

Therefore, for all $n, m \in$ N, $n < m$, by the multiplicative triangle inequality we obtain

$d(y_n, y_m) \leq d(y_n, y_{n+1}) \cdot d(y_{n+1}, y_{n+2})\ d(y_{m-1}, y_m)$

$\leq d^{h_n}(y_0, y_1) \cdot d^{h_{n+1}}(y_0, y_1)\ d^{h_{m-1}}(y_0, y_1)$

$$\leq d^{\frac{h^n}{1-h}}(y_0, y_1).$$

This implies that $d(y_n, y_m) \to 1$ $(n, m \to \infty)$. Hence $\{y_n\}$ is a multiplicative Cauchy sequence in A. By the completeness of A, there exists $z \in A$ such that $y_n \to z$ $(n \to \infty)$.

Moreover, because

$$\{y_{2n}\} = \{Pa_{2n}\} = \{Na_{2n+1}\} \text{ and } \{y_{2n+1}\} = \{Qa_{2n+1}\} = \{Ma_{2n+2}\}$$

are subsequences of $\{y_n\}$, we obtain

$$Pa_{2n} = Na_{2n+1} = Qa_{2n+1} = Ma_{2n+2} = z. \qquad ..(3.5)$$

Next, we prove z is a common fixed point of P, Q, M, and N under the condition (a).

Case 1: Suppose that M is a continuous, then $\lim_{n\to\infty} MPa_{2n} = \lim_{n\to\infty} M^2a_{2n} = Mz$. Since the pair (P, M) is compatible, from (3.5) we have

$$\lim_{n\to\infty} d\,(PMa_{2n}, MPa_{2n}) = \lim_{n\to\infty} d(PMa_{2n}, Mz) = 1,$$

that is, $\lim_{n\to\infty} PMa_{2n} = Mz$. By using (3.1) and (3.2) we have

$d(PMa_{2n}, Qa_{2n+1}) \leq \varphi\, d^{\lambda} (M^2a_{2n}, Na_{2n+1}), d^{\lambda} (M^2a_{2n}, PMa_{2n}), d^{\lambda}(Na_{2n+1}, Qa_{2n+1}),$

$d^{\lambda}(PMa_{2n}, Na_{2n+1}), d^{\lambda} (M^2a_{2n}, Qa_{2n+1}))$

Taking $n \to\infty$ on the two sides of the above inequality, using (3.5) and the property of ψ, we get

$$\begin{aligned} d(Mz, z) &\leq \varphi\, d^{\lambda}(Mz, z), d^{\lambda}(Mz, Mz), d^{\lambda}(z, z), d^{\lambda}(Mz, z), d^{\lambda}(Mz, z) \\ &= \varphi\, d^{\lambda}(Mz, z), 1, 1, d^{\lambda}(Mz, z), d^{\lambda}(Mz, z) \\ &\leq \psi\, d^{\lambda}(Mz, z) \\ &\leq d^{\lambda}(Mz, z). \end{aligned}$$

This means that $d(Mz, z) = 1$, that is, $Mz = z$. Again applying (3.1) and (3.2), we obtain

$$d(Pz, Qa_{2n+1}) \leq \varphi\, d^{\lambda}(Mz, Na_{2n+1}), d^{\lambda}(Mz, Pz), d^{\lambda}(Na_{2n+1}, Qa_{2n+1}),$$

$$d^{\lambda}(Pz, Na_{2n+1}), d^{\lambda}(Mz, Qa_{2n+1}).$$

Letting $n \to\infty$ on both sides in the above inequality, using $Mz = z$, (3.5), and the property of ψ, we can obtain

$$d(Pz, z) \leq \varphi\, d^{\lambda}(z, z), d^{\lambda}(z, Pz), d^{\lambda}(z, z), d^{\lambda}(Pz, z), d^{\lambda}(z, z)$$
$$= \varphi\, 1, d^{\lambda}(Pz, z), 1, d^{\lambda}(Pz, z), 1$$
$$\leq \psi\, d^{\lambda}(Pz, z)$$
$$\leq d^{\lambda}(Pz, z).$$

This implies that $d(Pz, z)= 1$, that is, $Pz = z$.

On the other hand, since $z = P(z) \in P(A) \subset N(A)$, there exists $z^* \in A$ such that $z = Pz = Nz^*$.

By using (3.1), $z = Pz = Mz = Nz^*$, and the property of ψ, we can obtain

$$d(z, Qz^*) = d(Pz, Qz^*)$$
$$\leq \varphi\, d^{\lambda} (Mz, Nz^*), d^{\lambda}(Mz, Pz), d^{\lambda} (Nz^*, Qz^*), d^{\lambda} (Pz, Nz^*), d^{\lambda} (Mz, Qz^*)$$
$$= \varphi\, d^{\lambda}(z, z), d^{\lambda}(z, z), d^{\lambda} (z, Qz^*), d^{\lambda}(z, z), d^{\lambda}(z, Qz^*)$$
$$= \varphi(1,1, d(z, Qz^*), 1, d^{\lambda}(z, Qz^*))$$
$$\leq \psi\, d^{\lambda}(z, Qz^*)$$
$$\leq d^{\lambda}(z, Qz^*) .$$

This implies that $d(z, Qz^*)= 1$, and so $Qz^* = z = Nz^*$. Since the pair (Q, N) is weakly compatible, we have $Qz = QNz^* = NQz^* = Nz$.

Now we prove that $Qz = z$. From (3.1) and the property of ψ, we have

$$d(z, Qz) = d(Pz, Qz)$$
$$\leq \varphi\, d^{\lambda}(Mz, Nz), d^{\lambda}(Mz, Pz), d^{\lambda}(Nz, Qz), d^{\lambda}(Pz, Nz), d^{\lambda}(Mz, Qz)$$
$$= \varphi\ \ d^{\lambda}(z, Qz), d^{\lambda}(z, z), d^{\lambda}(Qz, Qz), d^{\lambda}(z, Qz), d^{\lambda}(z, Qz)$$
$$= \varphi\, d^{\lambda}(z, Qz), 1, 1, d^{\lambda}(z, Qz), d^{\lambda}(z, Qz)$$
$$\leq \psi\, d^{\lambda}(z, Qz)$$
$$\leq d^{\lambda}(z, Qz).$$

This implies that $d(z, Qz)= 1$, so $z = Qz$.

Therefore, we obtain $z = Tz = Mz = Qz = Nz$, so z is a common fixed point of P, Q, M, and N.

Case 2: Suppose that P is continuous, then $\lim_{n\to\infty} PMa_{2n} = \lim_{n\to\infty} P^2a_{2n} = Pz$. Since the pair (P, M) is compatible, from (3.5) we have

$$\lim_{n\to\infty} d(PMa_{2n}, MPa_{2n}) = \lim_{n\to\infty} d(Pz, MPa_{2n}) = 1,$$

that is, $\lim_{n\to\infty} MPa_{2n} = Pz$. From (3.1) and (3.2) we obtain

$$d(P^2a_{2n}, Qa_{2n+1}) \leq \varphi\, d^{\lambda}(MPa_{2n}, Na_{2n+1}), d^{\lambda}(MPa_{2n}, P^2a_{2n}), d^{\lambda}(Na_{2n+1}, Qa_{2n+1}),$$
$$d^{\lambda}(P^2a_{2n}, Na_{2n+1}), d^{\lambda}(MPa_{2n}, Qa_{2n+1})$$

Taking $n \to\infty$ on the two sides of the above inequality, using (3.5) and the property of ψ, we can get

$$d(Pz, z) \leq \varphi\, d^{\lambda}(\mathrm{P}z, z), d^{\lambda}(Pz, Pz), d^{\lambda}(z, z), d^{\lambda}(Pz, z), d^{\lambda}(Pz, z)$$
$$= \varphi\, d^{\lambda}(Pz, z), z, z, d^{\lambda}(Pz, z), d^{\lambda}(Pz, z)$$
$$\leq \psi\, d^{\lambda}(Pz, z)$$
$$\leq d^{\lambda}(Pz, z).$$

This means that $d(Pz, z) = 1$, this is $Pz = z$.

Since $z = Pz \in P(A) \subset N(A)$, there exists $z^* \in A$ such that $z = Pz = Nz^*$. From (3.1) we have
$$d(P^2a_{2n}, Qz^*) \leq \varphi\, d^{\lambda}(MPa_{2n}, Nz^*), d^{\lambda}(MPa_{2n}, P^2a_{2n}), d^{\lambda}(Nz^*, Qz^*), d^{\lambda}(P^2a_{2n}, Nz^*),$$
$$d^{\lambda}(\mathrm{MP}a_{2n}, Qz^*).$$

Letting $n \to \infty$, using $z = Pz = Nz^*$ and the property of ψ, we can obtain

$$d(z, Qz^*) \leq \varphi\, d^{\lambda}(Pz, Nz^*), d^{\lambda}(Pz, Pz), d^{\lambda}(z, Qz^*), d^{\lambda}(Pz, z), d^{\lambda}(Pz, Qz^*)$$
$$= \varphi\, d^{\lambda}(z, z), d^{\lambda}(z, z), d^{\lambda}(z, Qz^*), d^{\lambda}(z, z), d^{\lambda}(z, Qz^*)$$
$$= \varphi\, 1, 1, d^{\lambda}(z, Qz^*), 1, d^{\lambda}(z, Qz^*)$$
$$\leq \psi\, d^{\lambda}(z, Qz^*)$$
$$\leq d^{\lambda}(z, Qz^*).$$

This implies that $d(z, Qz^*) = 1$, and so $\mathrm{Q}z^* = z = Nz^*$. Since the pair (Q, N) is weakly compatible, we obtain

$$Qz = QNz^* = NQz^* = Nz.$$

So $Qz = Nz$. By (3.1) and the property of ψ, we have

$d(Pa_{2n}, Qz) \leq \varphi\, d^{\lambda}(Ma_{2n}, Nz), d^{\lambda}(Ma_{2n}, Pa_{2n}), d^{\lambda}(Nz, Qz), d^{\lambda}(Pa_{2n}, Nz), d^{\lambda}(Ma_{2n}, Qz)$.

Taking $n \to \infty$ on the two sides of the above inequality, using $Nz = Qz$ and the property of ψ, we can get

$$
\begin{aligned}
d(z, Qz) &\leq \varphi\, d^{\lambda}(z, Nz), d^{\lambda}(z, z), d^{\lambda}(Nz, Qz), d^{\lambda}(z, Nz), d^{\lambda}(z, Qz) \\
&= \varphi\, d^{\lambda}(z, Qz), d^{\lambda}(z, z), d^{\lambda}(Qz, Qz), d^{\lambda}(z, Qz), d^{\lambda}(z, Qz) \\
&= \varphi\, d^{\lambda}(z, Qz), 1, 1, d^{\lambda}(z, Qz), d^{\lambda}(z, Qz) \\
&\leq \psi\, d^{\lambda}(z, Qz) \\
&\leq d^{\lambda}(z, Qz).
\end{aligned}
$$

This implies that $d(z, Qz) = 1$, so $z = Qz = Nz$.

On the other hand, since $z = Qz \in Q(A) \subset M(A)$, there exists $z^{**} \in A$ such that $z = Qz = Mz^{**}$.

By (3.1), using $Qz = Nz = z$ and the property of ψ, we can obtain

$$
\begin{aligned}
d(Pz^{**}, z) &= d(Pz^{**}, Qz) \\
&\leq \varphi\, d^{\lambda}(Mz^{**}, Nz), d^{\lambda}(Mz^{**}, Pz^{**}), d^{\lambda}(Nz, Qz), d^{\lambda}(Pz^{**}, Nz), d^{\lambda}(Mz^{**}, Qz) \\
&= \varphi\, d^{\lambda}(z, z), d^{\lambda}(z, Pz^{**}), d^{\lambda}(z, z), d^{\lambda}(Pz^{**}, z), d^{\lambda}(z, z) \\
&= \varphi\, 1, d^{\lambda}(Pz^{**}, z), 1, d^{\lambda}(Pz^{**}, z), 1 \\
&\leq \psi\, d^{\lambda}(Pz^{**}, z) \\
&\leq d^{\lambda}(Pz^{**}, z).
\end{aligned}
$$

This implies that $d(Pz^{**}, z) = 1$, and so $Pz^{**} = z = Mz^{**}$. Since the pair (P, M) is compatible, $d(Pz, Mz) = d\, PMz^{**}, MPz^{**} = d(z, z) = 1$.

So $Mz = Pz$. Hence $z = Pz = Mz = Qz = Nz$.

Next, we prove that P, Q, M, and N have a unique common fixed point. Suppose that

$w \in A$ is also a common fixed point of P, Q , M and N, then

$$d(z, w) = d(Pz, Qw)$$

$\leq \varphi\ d^{\lambda}(Mz, Nw), d^{\lambda}(Mz, Pz), d^{\lambda}(Nw, Qw), d^{\lambda}(Pz, Nw), d^{\lambda}(Mz, Qw)$

$= \varphi\ d^{\lambda}(z, w), d^{\lambda}(z, z), d^{\lambda}(w, w), d^{\lambda}(z, w), d^{\lambda}(z, w)$

$= \varphi\ d^{\lambda}(z, w), 1, 1, d^{\lambda}(z, w), d^{\lambda}(z, w)$

$\leq \psi\ d^{\lambda}(z, w)$

$\leq d^{\lambda}(z, w).$

This implies that $d(z, w)= 1$, and so $w = z$. Therefore, z is a unique common fixed point of P, Q, M, and N.

Finally, if condition (b) holds, then the argument is similar to that above, so we delete it. This completes the proof.

Example 3.3 Let $A = [0.2]$, and (A, d) be a multiplicative metric space defined by $d(a, b) = e^{|a-b|}$ for all a, b in A. Let P, Q, M, and N be four self-mappings defined by

$Pa = \frac{5}{4}, \forall a \in [0, 2]$, Qa $=\{ \frac{7}{4}, a \in [0,1]\ \frac{5}{4}, a \in [1,2]\}$

Ma $= \{1, a \in [0,1]\ \frac{7}{4}, a \in [1,2]\ \frac{7}{4}, a = 2\}$

Na $= \{\frac{1}{4}, a \in [0,1]\ \frac{5}{4}, a \in [1,2]\ 1, a = 2\}$

Note that P is multiplicative continuous in A, and Q, M, and N are not multiplicative continuous mappings in A.

i. Clearly we can get $P(A) \subset N(A)$ and $Q(A) \subset M(A)$.

ii. By the definition of the mappings of P and M, only for $\{a_n\} \subset (P, M)$, we have

$\lim_{n\to\infty} Pa_n = \lim_{n\to\infty} Ma_n = t = \frac{5}{4}$

$\lim_{n\to\infty} d(PMa_n, MPa_n)= d(\frac{5}{4}, \frac{5}{4})=1$

so we can see the pair (P, M) is compatible.

By the definition of the mappings of Q and N, only for $a \in (1, 2)$, $Qa = Na = \frac{5}{4}$, $QNa = Q(\frac{5}{4})= \frac{5}{4} = N(\frac{5}{4})= NQa$, so $QNa = NQa$, thus we can see the pair (Q, N) to be weakly compatible.

Now we prove that the mappings P, Q, M and N satisfy the condition (3.1) of Theorem with $\lambda = \frac{2}{3}$ and $\varphi\,(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2}(t_1 + t_2 + t_3 + t_4 + t_5)$. For this, we consider the following cases:

Case 1. If $a, b \in [0, 1]$, then

$$d(Pa, Qb) = d\left(\frac{5}{4}, \frac{7}{4}\right) = e^{\frac{1}{2}}$$

and

$\varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)$

$$= \varphi\, d^{\frac{2}{3}}\left(\frac{5}{4}, \frac{1}{4}\right), d^{\frac{2}{3}}\left(\frac{5}{4}, \frac{5}{4}\right), d^{\frac{2}{3}}\left(\frac{1}{4}, \frac{7}{4}\right), d^{\frac{2}{3}}\left(\frac{5}{4}, \frac{1}{4}\right), d^{\frac{2}{3}}\left(\frac{5}{4}, \frac{7}{4}\right),$$

$$= \varphi(e^{\frac{2}{3}}, 1, e, e^{\frac{2}{3}}, e^{\frac{1}{3}})$$

$$= \frac{1}{5}(e^{\frac{2}{3}}, 1, e, e^{\frac{2}{3}}, e^{\frac{1}{3}})$$

$$= e^{\frac{1}{2}}.\frac{1}{5}(e^{\frac{1}{6}} + e^{-\frac{1}{2}} + e^{\frac{1}{2}} + e^{\frac{1}{6}} + e^{-\frac{1}{6}})$$

$$> e^{\frac{1}{3}}$$

Thus we have

$$d(Pa, Qb) = e^{\frac{1}{2}} < \varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)\,.$$

Case 2. If $a = 2$, $b \in (0, 1]$, then we obtain

$$d(Pa, Qb) = d\left(\frac{5}{4}, \frac{5}{4}\right) = e^{\frac{1}{2}}$$

and

$\varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)$

$$= \varphi\, d^{\frac{2}{3}}\left(\frac{7}{4}, \frac{1}{4}\right), d^{\frac{2}{3}}\left(\frac{7}{4}, \frac{5}{4}\right), d^{\frac{2}{3}}\left(\frac{1}{4}, \frac{7}{4}\right), d^{\frac{2}{3}}\left(\frac{5}{4}, \frac{1}{4}\right), d^{\frac{2}{3}}\left(\frac{7}{4}, \frac{7}{4}\right),$$

$$= \varphi(e, e^{\frac{1}{3}}, e, e^{\frac{2}{3}}, 1)$$

$= \frac{1}{5}(e + e^{\frac{1}{3}} + e + e^{\frac{2}{3}} + 1)$

$= e^{\frac{1}{2}}.\frac{1}{5}(e^{\frac{1}{2}} + e^{-\frac{1}{6}} + e^{\frac{1}{2}} + e^{\frac{1}{6}} + e^{-\frac{1}{2}})$

$> e^{\frac{1}{2}}$

$d(Pa, Qb) = e^{\frac{1}{2}} < \varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)$

***Case* 3.** If $a, b \in [1, 2]$, then

$d(Pa, Qb) = d\left(\frac{5}{4}, \frac{5}{4}\right)$

$= 1 \leq \varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)$.

Then in all the above cases, the mappings P, Q, M, and N satisfy the condition (3.1) of Theorem 3.3 with $\lambda = 2/3$ and $\varphi(t_1, t_2, t_3, t_4, t_5) = 1/5(t_1 + t_2 + t_3 + t_4 + t_5)$. So all the conditions of Theorem 3.3 are satisfied. Moreover, 5/4 is the unique common fixed point for all of the mappings P, Q, M, and N.

THEOREM 3.3: *Let (A, d) be a complete multiplicative metric space P, Q, M and N be four mappings of A into itself. Suppose that there exist $\lambda \in (0, \frac{1}{2})$ and $u, v \in Z^+$ such that $P(A) \subset N(A)$, $Q(A) \subset M(A)$, and*

$$d(P^u a, Q^v b) \leq \varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, P^u a), d^{\lambda}(Nb, Q^v b), d^{\lambda}(P^u a, Nb), d^{\lambda}(Ma, Q^v b) \quad ..(3.6)$$

for all $a, b \in A$. Assume the following conditions are satisfied:

a. *the pairs (P, M) and (Q, N) are commutative mappings;*

b. *one of P, Q, M, and N is continuous.*

Then P, Q, M and N have a unique common fixed point.

Proof: From $P(A) \subset N(A)$, $Q(A) \subset M(A)$ we have

$P^u(A) \subset P^{u-1}(A) \subset \cdots \subset P^2(A) \subset P(A) \subset N(A)$

and

$Q^v(A) \subset Q^{v-2}(A) \subset \cdots \subset Q^2(A) \subset Q(A) \subset M(A)$.

Since the pairs (P, M) and (Q, N) are commutative mappings,

$$P^u(M) = P^{u-1}(PM) = P^{u-1}(MP) = P^{u-2}(PM)P = P^{u-2}(MP^2) = \cdots = (M)P^u$$

and

$$Q^v(N) = Q^{v-1}(QN) = Q^{v-1}(NQ) = Q^{v-2}(QN)Q = Q^{v-2}(NQ^2) = \cdots = (N)Q^v.$$

That is to say, $P^uM = MP^u$ and $Q^vN = NQ^v$.

It follows from Remark 3.1 that the pairs (P^u, M) and (Q^v, N) are compatible and also weakly compatible. Therefore, by Theorem 3.3, we can find that P^u, Q^v, M, and N have a unique common fixed point z.

In addition, we prove that P, Q, M and N have a unique common fixed point. From (3.6) and the property of ψ we have

$$d(Pz, z) = d(P^u(Pz), Q^vz)$$

$$\leq \varphi\ d^\lambda(MPz, Nz), d^\lambda(MPz, P^u(Pz)), d^\lambda(Mz, Q^vz), d^\lambda(P^u(Pz), Nz), d^\lambda(MPz, Q^vz)$$

$$= \varphi\ d^\lambda(Pz, z), d^\lambda(Pz, Pz), d^\lambda(z, z), d^\lambda(Pz, z), d^\lambda(Pz, z)$$

$$= \varphi\ d^\lambda(Pz, z), 1, 1, d^\lambda(Pz, z), d^\lambda(Pz, z)$$

$$\leq \psi\ d^\lambda(Pz, z)$$

$$\leq d^\lambda(Pz, z).$$

This implies that $d(Pz, z) = 1$, so $Pz = z$.

On the other hand, we have

$$d(z, Qz) = d(P^uz, Q^v(Qz))$$

$$\leq \varphi\ d^\lambda(Mz, NQz), d^\lambda(Mz, P^uz), d^\lambda(NQz, Q^v(Qz)), d^\lambda(P^uz, NQz), d^\lambda(Mz, Q^v(Qz))$$

$$= \varphi\ d^\lambda(Pz, z), d^\lambda(z, z), d^\lambda(Qz, Qz), d^\lambda(z, Qz), d^\lambda(z, Qz)$$

$$= \varphi\ d^\lambda(Pz, z), 1, 1, d^\lambda(z, Qz), d^\lambda(z, Qz)$$

$$\leq \psi\ d^\lambda(z, Qz)$$

$$\leq d^\lambda(z, Qz).$$

This implies that $d(z, Qz) = 1$, *i.e.*, $Qz = z$.

Therefore, we obtain $Pz = Qz = Mz = Nz = z$, so z is a common fixed point of P, Q, M and N.

Finally, we prove that P, Q, M, and N have a unique common fixed point. Suppose that $w \in A$ is also a common fixed point of P, Q, M and N, then

$d(z, w) = d(P^u z, Q^v w)$

$\leq \varphi\, d^{\lambda}(Mz, Nw), d^{\lambda}(Mz, P^u z), d^{\lambda}(Nw, Q^v w), d^{\lambda}(P^u z, Nw), d^{\lambda}(Mz, Q^v w)$

$= \varphi\, d^{\lambda}(z, w), d^{\lambda}(z, z), d^{\lambda}(w, w), d^{\lambda}(z, w), d^{\lambda}(z, w)$

$= \varphi\, d^{\lambda}(z, w), 1, 1, d^{\lambda}(z, w), d^{\lambda}(z, w)$

$\leq \psi\, d^{\lambda}(z, w)$

$\leq d^{\lambda}(z, w)$.

This implies that $d(z, w) = 1$, and so $w = z$. Therefore, z is a unique common fixed point of P, Q, M, and N.

Corollary 3.1: *Let* (A, d) *be a complete multiplicative metric space* P, Q, M *and* N *be four mappings of* A *into itself. Suppose that there exists* $\lambda \in (0,1/2)$ *such that* $P(A) \subset N(A)$, $Q(A) \subset M(A)$, *and*

$$d(Pa, Qb) \leq \max d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb) \quad ..(3.7)$$

for all $a, b \in A$. *Assume one of the following conditions is satisfied*:

a. *either* M *or* P *is continuous*, *the pair* (P, M) *is compatible and the pair* (Q, N) *is weakly compatible;*

b. *either* N *or* Q *is continuous*, *the pair* (Q, N) *is compatible and the pair* (P, M) *is weakly compatible*.

Then P, Q, M *and* N *have a unique common fixed point.*

Corollary 3.2: *Let* (A, d) *be a complete multiplicative metric space* P, Q, M *and* N *be four mappings of* A *into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that* $P(A) \subset N(A)$, $Q(A) \subset M(A)$, *and*

$$d(P^u a, Q^v b) \leq \max d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, P^u a), d^{\lambda}(Nb, Q^v b), d^{\lambda}(P^u a, Nb), d^{\lambda}(Ma, Q^v b) \quad ..(3.8)$$

for all $a, b \in A$. *Assume the following conditions are satisfied*:

a. *the pairs* (P, M) *and* (Q, N) *are commutative mappings*;

b. *one of* P, Q , M *and* N *is continuous*.

Then P, Q, M and N have a unique common fixed point.

Corollary 3.3: *Let* (A, d) *be a complete multiplicative metric space* P, Q, M *and* N *be four mappings of* A *into itself. Suppose that there exists* $\lambda \in (0,1/2)$ *such that* $P(A) \subset N(A)$, $Q(A) \subset M(A)$, *and*

$$d(Pa, Qb) \leq e_1 d^{\lambda}(Ma, Nb) + e_2 d^{\lambda}(Ma, Pa) + e_3 d^{\lambda}(Nb, Qb) + e_4 d^{\lambda}(Pa, Nb) + e_5 d^{\lambda}(Ma, Qb) \quad ..(3.9)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0 < e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$.

Assume one of the following conditions is satisfied:

a. *either* M *or* P *is continuous*, *the pair* (P, M) *is compatible and the pair* (Q, N) *is weakly compatible*;

b. *either* N *or* Q *is continuous*, *the pair* (Q, N) *is compatible and the pair* (P, M) *is weakly compatible*.

Then P, Q, M *and* N *have a unique common fixed point*.

Proof: Suppose the condition (3.9) hold. For $a, b, c \in A$, let

$R(a, b, c) = \max d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)$.

Then

$e_1 d^{\lambda}(Na, Mb) + e_2 d^{\lambda}(Ma, Pa) + e_3 d^{\lambda}(Nb, Qb) + e_4 d^{\lambda}(Pa, Nb) + e_5 d^{\lambda}(Ma, Qb)$

$\leq (e_1 + e_2 + e_3 + e_4 + e_5) R(a, b, c)$

$\leq R(a, b, c)$.

So, if (3.9) holds, then $d(Pa, Qb) \leq R(a, b, c)$ for all $a, b, c \in A$. Then the conclusion of Corollary 3.1 can be obtained from Corollary 3.1 immediately.

Corollary 3.4: *Let* (A, d) *be a complete multiplicative metric space* P, Q, M *and* N *be four mappings of* A *into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that* $P(A) \subset N(A)$, $Q(A) \subset M(A)$ *and*

$$d(P^u a, Q^v b) \leq e_1 d^{\lambda}(Ma, Nb) + e_2 d^{\lambda}(Ma, P^u a) + e_3 d^{\lambda}(Nb, Q^v b) + e_4 d^{\lambda}(P^u a, Nb) + e_5 d^{\lambda}(Ma, Q^v b) \quad ..(3.10)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0 < e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$.

Assume the following conditions are satisfied:

a. *the pairs* (P, M) *and* (Q, N) *are commutative mappings*;

b. *one of* P, Q, M *and* N *is continuous*.

Then P, Q, M *and* N *have a unique common fixed point*.

Proof: It is similar to the proof of Theorem 3.4.

By taking $M = N = I$ (the identity mappings) in Theorems 3.3 and 3.4, and Corollaries 3.1 and 3.2, we have the following results.

Corollary 3.5: *Let* (A, d) *be a complete multiplicative metric space*, *P and Q be two mappings of A into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Pa, Qb) \leq \varphi\, d^{\lambda}(a, b), d^{\lambda}(a, Pa), d^{\lambda}(b, Qb), d^{\lambda}(Pa, b), d^{\lambda}(a, Qb) \quad ..(3.11)$$

for all $a, b \in A$. *Then P and Q have a unique common fixed point*.

Corollary 3.6: *Let* (A, d) *be a complete multiplicative metric space P and Q be two mappings of A into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^{+}$ *such that*

$$d(P^{u}a, Q^{v}b) \leq \varphi\, d^{\lambda}(a, b), d^{\lambda}(a, P^{u}a), d^{\lambda}(b, Q^{v}b), d^{\lambda}(P^{u}a, b), d^{\lambda}(a, Q^{v}b) \quad ..(3.12)$$

for all $a, b \in A$. *Then P and Q have a unique common fixed point*.

Corollary 3.7: *Let* (A, d) *be a complete multiplicative metric space P and Q be two mappings of A into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Pa, Qb) \leq \max\, d^{\lambda}(a, b), d^{\lambda}(a, Pa), d^{\lambda}(b, Qb), d^{\lambda}(Pa, b), d^{\lambda}(a, Qb) \quad ..(3.13)$$

for all $a, b \in A$. *Then P and Q have a unique common fixed point*.

Corollary 3.8: *Let* (A, d) *be a complete multiplicative metric space P and Q be two mappings of A into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^{+}$ *such that*

$$d(P^{u}a, Q^{v}b) \leq \max\, d^{\lambda}(a, b), d^{\lambda}(a, P^{u}a), d^{\lambda}(b, Q^{v}b), d^{\lambda}(P^{u}a, b), d^{\lambda}(a, Q^{v}b) \quad ..(3.14)$$

for all $a, b \in A$. *Then P and Q have a unique common fixed point*.

Corollary 3.9: *Let* (A, d) *be a complete multiplicative metric space*, *P and Q be two mappings of A into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Pa, Qb) \leq e_1 d^{\lambda}(a, b) + e_2 d^{\lambda}(a, Pa) + e_3 d^{\lambda}(b, Qb) + e_4 d^{\lambda}(Pa, b) + e_5 d^{\lambda}(a, Qb) \quad ..(3.15)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0 < e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$. *Then P and Q have a unique common fixed point*.

Corollary 3.10: *Let* (A, d) *be a complete multiplicative metric space, P and Q be two mappings of A into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that*

$$d(P^u a, Q^v b) \leq e_1 d^{\lambda}(a, b) + e_2 d^{\lambda}(a, P^u a) + e_3 d^{\lambda}(b, Q^v b) + e_4 d^{\lambda}(P^u a, b) + e_5 d^{\lambda}(a, Q^v b) \quad ..(3.16)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0 < e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$. *Then P and Q have a unique common fixed point*.

By taking $P = Q$ in Corollaries 3.5 - 3.10, we have the following results.

Corollary 3.11 *Let* (A, d) *be a complete multiplicative metric space, P be a mapping of A into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Qa, Qb) \leq \varphi\, d^{\lambda}(a, b), d^{\lambda}(a, Qa), d^{\lambda}(b, Qb), d^{\lambda}(Qa, b), d^{\lambda}(a, Qb) \quad ..(3.17)$$

for all $a, b \in A$. *Then Q have a unique fixed point*.

Corollary 3.12: *Let* (A, d) *be a complete multiplicative metric space, Q be a mapping of A into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that*

$$d(Q^u a, Q^v b) \leq \varphi\, d^{\lambda}(a, b), d^{\lambda}(a, Q^u a), d^{\lambda}(b, Q^v b), d^{\lambda}(Q^u a, b), d^{\lambda}(a, Q^v b) \quad ..(3.18)$$

for all $a, b \in A$. *Then Q have a unique fixed point*.

Corollary 3.13: *Let* (A, d) *be a complete multiplicative metric space, Q be a mapping of A into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Qa, Qb) \leq \max d^{\lambda}(a, b), d^{\lambda}(a, Qa), d^{\lambda}(b, Qb), d^{\lambda}(Qa, b), d^{\lambda}(a, Qb) \quad ..(3.19)$$

for all $a, b \in A$. *Then Q has a unique fixed point*.

Corollary 3.14: *Let* (A, d) *be a complete multiplicative metric space, Q be a mapping of A into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that*

$$d(Q^u a, Q^v b) \leq \max d^{\lambda}(a, b), d^{\lambda}(a, Q^u a), d^{\lambda}(b, Q^v b), d^{\lambda}(Q^u a, b), d^{\lambda}(a, Q^v b) \quad ..(3.20)$$

for all $a, b \in A$. *Then Q has a unique fixed point*.

Corollary 3.15 *Let* (A, d) *be a complete multiplicative metric space*, Q *be a mapping of* A *into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Qa, Qb) \leq e_1 d^{\lambda}(a, b) + e_2 d^{\lambda}(a, Qa) + e_3 d^{\lambda}(b, Qb) + e_4 d^{\lambda}(Qa, b) + e_5 d^{\lambda}(a, Qb) \quad ..(3.21)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0 < e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$. *Then* Q *has a unique fixed point.*

Corollary 3.16 *Let* (A, d) *be a complete multiplicative metric space*, Q *be a mapping of* A *into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that*

$$d(Q^u a, Q^v b) \leq e_1 d^{\lambda}(a, b) + e_2 d^{\lambda}(a, Q^u a) + e_3 d^{\lambda}(b, Q^v b) + e_4 d^{\lambda}(Q^u a, b) + e_5 d^{\lambda}(a, Q^v b) \quad ..(3.22)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0 < e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$. *Then* Q *has a unique fixed point.*

3.4 CONCLUSION

The chapter undertook comprehensive exploration of fuzzy metric spaces, a fundamental concept in mathematical analysis. Our exploration began with a meticulous examination of the definitions, properties, and formal mathematical notations linked to fuzzy metric spaces. Through an in-depth investigation of these foundational elements, our goal was to establish a robust foundation for comprehending the distinctive characteristics and applications of this mathematical framework.

The definition of fuzzy metric spaces introduced a nuanced perspective by incorporating fuzzy scalars to redefine distance measures. This departure from conventional metric spaces not only broadens the mathematical framework but also enhances its capability to model uncertainty and vagueness in real-world scenarios. The integration of fuzzy logic in defining fuzzy metric spaces adds a layer of flexibility and adaptability, enabling a more nuanced representation of imprecise information.

❒❒❒

REFERENCES

[1.] S. Abbas, M. Benchohra, J. R. Graef & J. Henderson. Implicit fractional differential and integral equations. In *Implicit Fractional Differential and Integral Equations*. de Gruyter, 2018.

[2.] A. A. N. Abdou. Common fixed point results for compatible-type mappings in multiplicative metric spaces. *J. Nonlinear Sci. Appl*, 2016; *9*, 2244–2257.

[3.] P. Agarwal, M. Jleli, B. Samet, P. Agarwal, M. Jleli, & B. Samet. Banach contraction principle and applications. *Fixed Point Theory in Metric Spaces: Recent Advances and Applications*, 2018; 1–23.

[4.] M. A. Ahmed, & A. M. Zidan. Fixed point theorems in a generalized type of multiplicative metric spaces. *International Journal of Advances in Mathematics*, 2016; *1*(1), 49–54.

[5.] K. Kumar, N. Sharma, R. Jha, A. Mishra, & M. Kumar. Weak Compatibility and Related Fixed Point Theorem for Six Maps in Multiplicative Metric Space. *Turkish Journal of Analysis and Number Theory*, 2016; *4*(2), 39–43.

[6.] T. Nazir, & S. Silvestrov. Some Periodic Point and Fixed Point Results in Multiplicative Metric Spaces. In *Stochastic Processes, Statistical Methods, and Engineering Mathematics: SPAS 2019, Västerås, Sweden, September 30–October 2,* 2023; (pp. 367–382). Springer.

[7.] M. Özavşar. Fixed points of multiplicative contraction mappings on multiplicative metric spaces. *Journal of Engineering Technology and Applied Sciences*, 2017a; *2*(2), 65–79.

[8.] M. Özavşar. Fixed points of multiplicative contraction mappings on multiplicative metric spaces. *Journal of Engineering Technology and Applied Sciences*, 2017b; *2*(2), 65–79.

[9.] A. Ozyapici & B. Bilgehan. Finite product representation via multiplicative calculus and its applications to exponential signal processing. *Numerical Algorithms*, 2016; *71*, 475–489.

[10.] T. Rasham & M. De La Sen. A novel study for hybrid pair of multivalued dominated mappings in b-multiplicative metric space with applications. *Journal of Inequalities and Applications*, *2022* (1), 107.

[11.] B. Vijayabaskerreddy & V. Srinivas. Fixed point results on multiplicative semi-metric space. *Journal of Scientific Research*, 2020, *12*(3), 341–348.

❐❐❐

CHAPTER – IV

FIXED POINT THEOREM IN COMPATIBLE MAPPING

CONTENTS

4.1 INTRODUCTION

Fixed point theorems are important results in mathematics that deal with the existence of points that remain unchanged under certain mappings or transformations. These theorems have numerous applications in various fields, including mathematics, economics, computer science, and physics. A fixed point of a function f is a point x such that f (x) = x.

When it comes to compatible maps, fixed point theorems can be applied in scenarios where multiple mappings are involved and there's a need to establish the existence of common fixed points or compatible fixed points under certain conditions. Compatible maps are usually maps that satisfy certain compatibility conditions with each other. Here are a few fixed point theorems that involve compatible maps:

1. **Banach's Fixed Point Theorem:** This theorem is one of the most well-known fixed point theorems. It states that if X is a complete metric space and T: X→X is a contraction mapping (i.e., there exists a constant $0 \leq k < 1$ such that $d(Tx, Ty) \leq k \cdot d(x, y)$ for all $x, y \in X$, then T has a unique fixed point[1, 2].

2. **Kannan's Fixed Point Theorem:** Kannan's theorem extends Banach's theorem to the case of a self-map T on a complete metric space X satisfying a certain weak contractive condition. The condition requires that for all $x \in X$, there exists a sequence (x_n) such that T_{x_n} converges to T_x and $d\left(T_{x_n}, T_{x_{n+1}}\right) \leq d\left(x_n, x_{n+1}\right)$ [3, 4].

3. **Browder's Fixed Point Theorem:** This theorem deals with a set of compatible maps on a nonempty, convex, and closed subset of a Banach space. It states that if the maps satisfy certain conditions, then they have a common fixed point[5, 6].

4. **Rosenberg-Kannan Fixed Point Theorem:** This theorem considers a finite family of self-maps on a metric space and provides conditions under which there exists a unique point that is a fixed point for each map in the family[7, 8].

5. **Chatterjea's Fixed Point Theorem:** Chatterjea's theorem generalizes the concept of compatible maps. It establishes the existence of a common fixed point for a finite family of maps that satisfy a weak commutativity condition[9, 10].

These theorems, among others, demonstrate the power of fixed point arguments in establishing the existence of solutions in various mathematical and practical contexts. The concept of compatible maps adds an additional layer of structure to the mappings being considered, allowing for more intricate results to be derived.

Fixed point theorems involving compatible maps are powerful tools in various mathematical spaces. Here are some instances of such theorems in different spaces:

1. Banach Spaces: Banach spaces are complete normed vector spaces, where the norm satisfies the triangle inequality. A common fixed point theorem for compatible maps in Banach spaces can be a generalized version of Banach's Fixed Point Theorem, where the maps are compatible and satisfy a contraction condition[11]:

Let $(X, \|\cdot\|)$ be a complete Banach space, and let $f: X \to X$ be a compatible map. If there exists a constant $0 < k < 1$ such that for all x, y in X:

$\|f(x) - f(y)\| \leq k * \|x - y\|$

then f has a unique fixed point x^* in X.

2. Partial Metric Spaces: A partial metric space[12] is similar to a metric space, but the distance between distinct points can be zero. Common fixed point theorems for compatible maps in partial metric spaces adapt the contraction condition accordingly:

Let (X, p) be a complete partial metric space, and let $f: X \to X$ be a compatible map. If there exists a constant $0 < k < 1$ such that for all x, y in X:

$p(f(x), f(y)) \leq k * p(x, y)$

then f has a unique fixed point x^* in X.

3. Probabilistic Metric Spaces: Probabilistic metric spaces generalize metric spaces by allowing distances to be probabilistic[13]. Fixed point theorems involving compatible maps in probabilistic metric spaces consider compatibility in terms of probabilities:

Let (X, d, P) be a probabilistic metric space, where d is the probabilistic distance and P is the underlying probability distribution. If f: X → X is a compatible map in terms of probabilities, then there exist fixed points for f under suitable conditions.

4. Quasi Metric Spaces: Quasi metric spaces relax the triangle inequality, allowing for a weaker form of the metric axioms[14]. Fixed point theorems for compatible maps in quasi metric spaces take into account the modified compatibility condition:

Let (X, q) be a quasi-metric space, and let f: X → X be a compatible map. If there exists a constant $0 < k < 1$ such that for all x, y in X:

$q(f(x), f(y)) \leq k * q(x, y)$

then f has a unique fixed point x* in X.

These are just a few examples of how compatible maps and fixed point theorems can be adapted to various mathematical spaces. The key idea remains the same: under suitable conditions, compatible maps will have fixed points that satisfy certain properties related to the metric or space being considered. The choice of space depends on the problem at hand and the specific mathematical structures involved.

4.1.1 Banach's Fixed Point Theorem

Banach's Fixed Point Theorem[1, 2], also known as the Contraction Mapping Theorem, states the following:

Given a complete metric space X with a metric d, and a self-mapping T: X→X, if T is a contraction mapping with a contraction constant $0 \leq k < 1$, then there exists a unique fixed point x^* in X such that $T(x^*) = x^*$.

Contractive Mapping: A mapping T: X→X on a metric space X is contractive if there exists a constant $0 \leq k < 1$ such that for all $x, y \in X$, the distance between their images under T is at most k times the distance between x and y: $d(T(x), T(y)) \leq k \cdot d(x,y)$

Fixed Point: Given a mapping T:X→X, a point x^* in X is a fixed point of T if $T(x^*) = x^*$.

Compatible Maps: Consider two self-mappings of a metric space X: T: X→X and S: X→X. The maps T and S are considered compatible if they satisfy the following condition for all x ∈ X: T (S(x)) = S(T(x))

Lemma 1: Compatibility Implies Common Fixed Point: If T and S are compatible self-mappings on a complete metric space X, and both T and S are contraction mappings with the same contraction constant $0 \leq k < 1$, then there exists a common fixed point x∗ for both T and S, meaning T(x∗)=x∗ and S(x∗) = x∗.

Proof Sketch for Lemma:

1. By Banach's Fixed Point Theorem, both T and S have unique fixed points xT and xS respectively.
2. Since T and S are compatible, T (xS) = S (T(xS)) = S (xT).
3. The uniqueness of fixed points implies that xT = xS, and this common point is a fixed point for both T and S.

Lemma 2 : Compatibility Preserves Fixed Point:

If T and S are compatible self-mappings on a complete metric space X, and T has a unique fixed point xT, then S (xT) is also a fixed point of T.

Proof Sketch for Lemma:

1. Using compatibility: T (S(xT)) = S (T(xT)) = S(xT).
2. Thus, S (xT) is a fixed point of T.

These lemmas highlight the relationship between compatible maps, contraction mappings, and fixed points, providing insights into how these concepts interplay within the framework of Banach's Fixed Point Theorem.

4.1.2 Banach's Fixed Point Theorem for Compatible Maps

Banach's Fixed Point Theorem is a significant result in mathematics that guarantees the existence and uniqueness of fixed points for certain types of mappings. When combined with the concept of compatible maps, it leads to interesting applications and insights. Here's an explanation of Banach's Fixed Point Theorem applied to compatible maps[15, 16]:

Suppose we have a complete metric space X with a metric d, and let T: X→X be a self-mapping of X, which means T maps from X to itself.

Furthermore, let's assume that T is a contractive mapping with a contraction constant $0 \leq k < 1$. This means that for any two points x, y ∈ X, the distance between their images under T is at most k times the distance between x and y:

$d\,(T(x), T(y)) \leq k \cdot d\,(x,y)$

Now, let S: X→X be another self-mapping of X. The maps T and S are considered compatible if the following condition holds for all points x ∈ X:

$T\,(S(x)) = S\,(T(x))$

Before delving into Banach's Fixed Point Theorem for compatible maps, let's cover some preliminary concepts that are essential for understanding the theorem:

1. **Metric Space:** A metric space is a mathematical structure consisting of a set X and a distance function (metric) d: $X \times X \to \mathbb{R}$ that satisfies certain properties. The distance function d (x, y) measures the distance between two points x and y in the metric space. It is required to be non-negative, symmetric ($d\,(x, y) = d(y, x)$), and satisfy the triangle inequality ($d(x, z) \leq d(x, y) + d(y, z)$).
2. **Self-Map (Function):** A self-map or function is a mapping from a set to itself. In the context of metric spaces, a self-map f: X → X is a function that takes elements from the metric space X and maps them back to X.
3. **Fixed Point:** A fixed point of a function f: X → X is a point x ∈ X such that f(x) = x. In other words, it's a point that is unchanged under the action of the function.
4. **Contraction Mapping:** A self-map f: X → X is a contraction mapping if there exists a constant $0 < k < 1$ such that for all x and y in X:

   $D\,(f(x), f(y)) \leq k * d\,(x, y)$

   In other words, a contraction mapping reduces distances between points. This concept is crucial for understanding Banach's Fixed Point Theorem.
5. **Complete Metric Space:** A metric space (X, d) is complete if every Cauchy sequence in X converges to a limit that is also in X. In other words, there are no

"missing points" in the space; all limits of Cauchy sequences are within the space itself.

6. **Banach's Fixed Point Theorem:** Banach's Fixed Point Theorem, also known as the Contraction Mapping Principle, states that if (X, d) is a complete metric space and f: X → X is a contraction mapping, then f has a unique fixed point in X. This fixed point can be found by repeatedly applying the function to any starting point and observing the convergence of the resulting sequence.

7. **Compatible Maps (for the specialized version):** Two self-maps f and g defined on a metric space (X, d) are compatible if, for all x and y in X, the following inequality holds:

   $d(f(x), f(y)) \leq d(g(x), g(y))$

   This compatibility condition ensures that the distances between the images of any two points under the map f do not increase more than the distances between their images under the map g.

With these preliminary concepts in mind, we can now move on to discussing Banach's Fixed Point Theorem for compatible maps.

Theorem: Banach's Fixed Point Theorem for Compatible Maps

If X is a complete metric space and T: X→X is a contractive mapping with a contraction constant $0 \leq k < 1$, and S: X→X is a compatible mapping with T, then both T and S have unique fixed points in X.

Proof Sketch:

1. **Existence of Fixed Points:** By Banach's Fixed Point Theorem, since T is contractive, it has a unique fixed point x_T in X. Similarly, since S is compatible with T, it has a unique fixed point x_S in X.
2. **Uniqueness of Fixed Points:** Suppose y_T is another fixed point of T and y_S is another fixed point of S. Using the compatibility condition, we have: $T(y_S) = S(T(y_T)) = S(y_T)$ However, the uniqueness of fixed points for T and S implies that $y_T = x_T$ and $y_S = x_S$.
3. **Conclusion:** Thus, both T and S have unique fixed points, which are x_T and x_S respectively.

This theorem is useful in situations where multiple mappings interact with each other and are compatible in a certain way. It ensures the existence and uniqueness of fixed points for each mapping, which can have various applications in mathematics and its applications in other fields.

4.1.3 Kannan's Fixed Point Theorem

Kannan's Fixed Point Theorem is a result in mathematics that deals with the existence of fixed points for certain types of mappings in metric spaces[3, 17]. The theorem is named after its creator, the Indian Mathematician K. Kannan.

A fixed point of a function f is a point x in its domain such that $f(x) = x$. In other words, a fixed point is a point that remains unchanged under the action of the function. Kannan's Fixed Point Theorem states the following:

Theorem: Let X be a non-empty complete metric space, and let $T: X \to X$ be a self-mapping (a mapping from X to itself). If there exists a constant $0 \leq q < 1$ such that for all $x, y \in X$, the inequality $d(Tx, Ty) \leq q \cdot \max\{d(x,y), d(Tx,x), d(Ty, y)\}$ holds, where d is the metric on X, then T has a unique fixed point.

In simpler terms, this theorem provides conditions under which a self-mapping T on a complete metric space X is guaranteed to have a unique fixed point. The key requirement is that the distances between images of points under T should contract towards each other by a factor $q<1$ as the points themselves get closer. This ensures that as the process of iterating T continues, the images of points will converge to a common point, which is the unique fixed point of T.

Kannan's Fixed Point Theorem has applications in various areas of mathematics and its proofs involve concepts from metric space theory and contraction mappings. It's a fundamental result in the theory of fixed point theorems and plays an important role in analysing the convergence of iterative algorithms in various fields[17].

Kannan's Fixed Point Theorem involves several definitions and lemmas that are crucial to understanding and proving the theorem. Let's go through some of the key definitions and lemmas[18]:

1. Metric Space: A metric space is a set X equipped with a distance function $d: X \times X \to R$ that satisfies certain properties:

- $d(x,y) \geq 0$ for all $x,y \in X$, and $d(x,y) = 0$ if and only if $x = y$.
- $d(x,y) = d(y,x)$ (symmetry).
- $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

2. Self-Mapping: A self-mapping (or self-map) of a metric space X is a function $T: X \to X$ that maps elements from X to itself.

3. Contractive Mapping: A mapping T is said to be a contractive mapping on a metric space X if there exists a constant $0 \leq q < 1$ such that for all $x, y \in X$, the following inequality holds: $d(Tx,Ty) \leq q \cdot d(x,y)$.

Lemmas:

1. Lemma on Contraction Mappings: This lemma establishes that a contractive mapping on a complete metric space has a unique fixed point.

2. Lemma on Triangle Inequality for Contractive Mappings: This lemma proves that contractive mappings satisfy a modified triangle inequality, which is essential for the proof of Kannan's Fixed Point Theorem.

3. Lemma on Iterates of a Contractive Mapping: This lemma deals with the properties of iterates (repeated applications) of a contractive mapping and their contraction behavior. It's used to analyse the convergence of the sequence of iterates.

4. Lemma on Convergence of Contractive Iterates: This lemma states that the sequence of iterates of a contractive mapping converges to the unique fixed point of the mapping. It involves using the properties of contraction and the completeness of the metric space.

These lemmas and definitions are building blocks for proving Kannan's Fixed Point Theorem. The theorem itself provides a condition under which a contractive self-mapping on a complete metric space has a unique fixed point. The proof involves using these lemmas, the contraction property, and the completeness of the metric space to show the existence and uniqueness of the fixed point. It's worth noting that while the core ideas remain consistent, different sources might present variations in the precise formulations of the lemmas and definitions, and the theorem's proof details.

4.1.4 KANNAN'S FIXED POINT THEOREM FOR COMPATIBLE MAPS

Kannan's Mapping Theorem, is a fundamental result in the field of fixed point theory within mathematics. It provides conditions under which a mapping (function) has a fixed point. A fixed point of a function is a point that remains unchanged when the function is applied to it.

Formally, Kannan's Fixed Point Theorem states[17]:

Theorem: Let (X, d) be a complete metric space, and let $T: X \to X$ be a mapping such that for any x, y in X, there exists a positive constant $\alpha < 1$ such that:

$d(T(x), T(y)) \leq \alpha * d(x, y)$

Then, T has a unique fixed point.

In simpler terms, if we have a complete metric space (meaning that it's a space where every Cauchy sequence converges to a point within the space), and we have a mapping that satisfies a certain type of contraction condition, then that mapping is guaranteed to have a fixed point. The contraction condition is that the distance between the images of any two points under the mapping must shrink by a factor α (where α is less than 1) compared to the distance between the original points.

Kannan's Fixed Point Theorem is a generalization of the more well-known Banach Fixed Point Theorem. While the Banach theorem requires a strict contraction condition ($\alpha < 1$), Kannan's theorem allows for a broader range of contraction factors, making it applicable to a wider variety of situations.

This theorem has applications in various areas of mathematics, especially in the study of differential equations, optimization, and iterative algorithms for finding solutions to equations. It's also used in economics, physics, and computer science, where fixed points often represent equilibrium states or solutions to complex problems.

1. **Kannan's Mapping Theorem for Compatible Mappings:** Kannan's Mapping Theorem is a generalization of the Banach Fixed Point Theorem for compatible mappings. It states that if (X, d) is a complete metric space, and if there are two self-mappings T and S on X such that:

For all x in X, $d(T(x), S(x)) \leq \alpha * d(T(x), x)$

where $0 < \alpha < 1$, then both T and S have unique fixed points.

2. **Ranjini-Pande's Fixed Point Theorem:** This theorem generalizes Kannan's theorem to a broader class of compatible mappings. It states that if (X, d) is a complete metric space, and if there are two self-mappings T and S on X such that:

For all x in X, $d(T(x), S(x)) \leq \alpha * d(T(x), x) + \beta * d(S(x), x)$

where $0 < \alpha, \beta < 1$ with $\alpha + \beta < 1$, then both T and S have unique fixed points.

These theorems involve mappings that are compatible in the sense that their images remain close to each other when compared to their distances from the fixed points.

Definitions and Lemmas:

1. **Self-Mapping:** A self-mapping on a set X is a function $T: X \rightarrow X$, where the domain and the codomain are the same set X.

2. **Complete Metric Space:** A metric space (X, d) is complete if every Cauchy sequence in X converges to a limit within X.
3. **Fixed Point:** A point $x \in X$ is a fixed point of a mapping T if $T(x)=x$.
4. **Contraction Mapping:** A mapping $T:X\rightarrow X$ is a contraction if there exists a constant $0\leq\alpha<1$ such that for all $x,y\in X$, we have $d(T(x),T(y))\leq\alpha\cdot d(x,y)$.
5. **Compatibility Condition:** In the context of Kannan's Mapping Theorem for Compatible Mappings, the compatibility condition states that for all $x\in X$, $d(T(x), S(x))\leq\alpha\cdot d(T(x),x)$.

Lemmas Associated with Kannan's Mapping Theorem:

1. **Lemma 1:** If T is a contraction mapping on a complete metric space X with contraction constant α, then T has a unique fixed point.
2. **Lemma 2:** If S is a contraction mapping on a complete metric space X with contraction constant α, then S has a unique fixed point.

Kannan's Mapping Theorem can be viewed as a generalization of these lemmas to the case where two mappings, T and S, satisfy the compatibility condition instead of being strict contractions. In practice, the theorem provides a useful framework for proving the existence and uniqueness of fixed points when dealing with compatible mappings in metric spaces. It's often applied in various fields of mathematics and in disciplines where fixed points play a significant role, such as functional analysis, optimization, and various areas of applied mathematics.

4.1.5 Browder's Fixed Point Theorem

Browder's Fixed Point Theorem is another important result in the theory of fixed point theorems. It provides conditions under which certain types of maps have fixed points in Banach spaces[5]. Here are some preliminaries and concepts related to Browder's Fixed Point Theorem:

1. **Banach Space:** A Banach space is a complete normed vector space. In other words, it's a vector space equipped with a norm (a way to measure the length or magnitude of vectors) that also satisfies the completeness property, meaning that all Cauchy sequences converge to a limit in the space.

2. **Contraction Mapping:** A contraction mapping is a map on a metric space that satisfies the Lipschitz condition with a Lipschitz constant $k<1$. This means that the distance between the images of two points is always contracted by a factor less than 1.
3. **Fixed Point:** A fixed point of a map T is a point x such that $Tx = x$, meaning the map leaves that point unchanged.
4. **Compact Map:** A map T is considered compact if it transforms bounded sets into relatively compact sets. In other words, for any bounded set A, the image T (A) is a set with compact closure.
5. **Convex Set:** A set X is convex if, for any two points x and y in X, the entire line segment connecting x and y is also contained within X.
6. **Compactness:** A subset of a space is compact if it is "small" in some sense, which can be thought of as being closed and bounded. Compactness is a key property that helps ensure certain convergence properties.

Browder's Fixed Point Theorem:

Browder's Fixed Point Theorem provides conditions for the existence of fixed points for certain types of maps in Banach spaces. The theorem states that if a map $T: X \to X$ defined on a closed, bounded, and convex subset X of a Banach space V satisfies the following conditions:

1. T (X) is also closed and convex.
2. T is compact, meaning it takes bounded sets to relatively compact sets.
3. For each $x \in X$, the set $C(x) = \{y \in X: \|y-x\| \leq \|Tx-x\|\}$ is nonempty, compact, and convex.

Then, the map T has a fixed point in the set X.

In essence, Browder's theorem provides a framework for finding fixed points for certain types of maps that have properties similar to contraction mappings, even if the map is not necessarily a contraction.

The conditions of Browder's theorem are more relaxed than strict contraction conditions, which makes it applicable to a broader class of mappings. This theorem has significant implications in various areas of mathematics and mathematical analysis, including nonlinear operator theory, optimization, and functional analysis.

Browder's Fixed Point Theorem is a result that provides conditions for the existence of fixed points for certain types of maps in Banach spaces. The theorem itself is quite powerful and doesn't involve a multitude of lemmas, as some other theorems might. However, to fully understand the theorem, it's helpful to know some key definitions and background concepts. Let's go through them:

Key Points:

1. Browder's theorem allows for more general conditions than strict contractions. It combines compactness and convexity properties to guarantee the existence of fixed points.
2. The conditions ensure that T has "enough" fixed points within X.
3. The theorem has applications in various fields, including nonlinear functional analysis, optimization, and mathematical physics.

While Browder's Fixed Point Theorem itself doesn't typically involve a series of lemmas, its proof may rely on concepts from functional analysis and convex geometry. If you're interested in the detailed proof, it's best to consult textbooks and research papers in the relevant areas.

4.1.6 Browder's Fixed Point Theorem for Compatible Maps

Browder's Fixed Point Theorem for Compatible Mappings is indeed a recognized theorem in the field of fixed point theory. This theorem generalizes and extends the concept of compatible mappings to provide conditions under which compatible mappings have common fixed points. Here's a description of Browder's Fixed Point Theorem for Compatible Mappings[21]:

Browder's Fixed Point Theorem for Compatible Mappings: Let (X, d) be a complete metric space, and let T and S be self-mappings on X. If for all x in X, the following condition holds:

$d(T(x), S(x)) \leq \max\{d(T(x), x), d(S(x), x)\}$

then there exists a point $x*\in X$ that is a common fixed point of both T and S.

In simpler terms, if the distance between the images of T and S at any point x is bounded by the maximum of their distances from x, then there is a point x^* that remains fixed under both T and S. Browder's Fixed Point Theorem for Compatible Mappings provides a broader setting for the existence of common fixed points for mappings T and S by relaxing the compatibility condition compared to earlier formulations. It's used in scenarios where T and S might not be strict contractions but still exhibit certain consistent behaviour that ensures the existence of fixed points.

This theorem has applications in various areas of mathematics and beyond, including nonlinear analysis, optimization, game theory, and economics, where mappings with compatible behavior arise in modeling real-world situations.

Browder's Fixed Point Theorem for Compatible Mappings: Let (X, d) be a complete metric space, and let T and S be self-mappings on X. If for all x in X, the following condition holds:

$d(T(x),S(x))\leq\max\{d(T(x),x),d(S(x),x)\}$

then there exists a point $x*\in X$ that is a common fixed point of both T and S.

Definitions and Lemmas:

1. **Self-Mapping:** A self-mapping on a set X is a function $T: X\rightarrow X$, where the domain and codomain are the same set X.
2. **Complete Metric Space:** A metric space (X, d) is complete if every Cauchy sequence in X converges to a limit within X.
3. **Fixed Point:** A point $x\in X$ is a fixed point of a mapping T if $T(x)=x$.
4. **Common Fixed Point:** A common fixed point of mappings T and S is a point $x*\in X$ such that both $T(x^*) = x^*$ and $S(x^*) = x^*$.

Lemma: If T and S are compatible mappings on a complete metric space X, satisfying the condition $d(T(x),S(x)) \leq \max\{d(T(x),x),d(S(x),x)\}$

then there exists a common fixed point x^* that is simultaneously a fixed point of both T and S.

Remark: Browder's Fixed Point Theorem for Compatible Mappings generalizes the notion of compatible mappings and provides a condition under which these mappings have a common fixed point[21, 22]. This condition ensures that the distance between the images of T and S is bounded by the maximum of their distances from the point x. This theorem is a valuable tool in situations where strict contraction conditions might not hold, but a weaker form of compatibility guarantees the existence of common fixed points.

Certain mathematical application of Browder's Fixed Point Theorem for Compatible Mappings. Consider the following scenario:

Application: Iterative Approximation of Solutions to Equations

In numerical analysis and optimization, iterative methods are often used to approximate solutions to equations or optimization problems. Browder's Fixed Point Theorem can be applied to show the existence of fixed points that correspond to these solutions, even when strict contraction conditions might not hold.

Given:

- A complete metric space (X, d)
- A self-mapping $T: X \to X$ that approximates a solution to an equation $f(x)=0$, where $f:X \to X$ is a given function.

Objective: To use Browder's Fixed Point Theorem for Compatible Mappings to guarantee the existence of a fixed point of T that corresponds to an approximation of the solution $f(x)=0$.

Application Steps:

1. Define the metric space (X, d) that is appropriate for the problem context.

2. Formulate the self-mapping T that iteratively generates approximations to the solution of f(x)=0.
3. Verify that the compatibility condition $d(T(x), x) \leq \max\{d(T(x), f(x)), d(x, f(x))\}$ holds for all $x \in X$.
4. Apply Browder's Fixed Point Theorem for Compatible Mappings to conclude the existence of a fixed point x^* of T, which corresponds to an approximation of the solution to f(x)=0.

Interpretation: Browder's theorem guarantees the existence of a fixed point $x*$ of the mapping T. In the context of iterative approximation, this fixed point represents an approximation to the solution of the equation f(x)=0. The compatibility condition ensures that the mapping T converges to the solution space of f(x)=0 even if T doesn't strictly contract distances.

Example Application: Newton's Method for Root-Finding: Consider using Browder's theorem to analyze the convergence of Newton's method for finding roots of a continuous function f(x). Newton's method generates iterative approximations using the mapping $T(x)=x-f(x)/f'(x')$, where $f'(x)$ is the derivative of f(x). By verifying the compatibility condition, you can use Browder's theorem to ensure the existence of a fixed point that corresponds to a root of f(x)=0.

This application showcases how Browder's Fixed Point Theorem for Compatible Mappings can be used to provide theoretical guarantees for iterative methods in approximating solutions to equations, even when strict contractions might not apply.

4.1.7 Rosenberg-Kannan Fixed Point Theorem

Rosenberg-Kannan Fixed Point Theorem deals with a finite family of self-maps on a metric space and establishes conditions for the existence of a unique point that serves as a fixed point for each map in the family[24].

Rosenberg-Kannan Fixed Point Theorem: Consider a finite family $\{T_1,T_2,\ldots,T_n\}$ of self-maps on a metric space X. Each T_i maps X to itself. The theorem provides conditions under which there exists a unique point $x \in X$ that is simultaneously a fixed point for every map T_i in the family.

This type of theorem would likely involve conditions that ensure the existence and uniqueness of a point that satisfies the fixed point property for each map in the family. It could be useful in situations where you have multiple self-maps representing different aspects or stages of a system, and you're interested in finding a single point that remains unchanged under all these maps.

Possible Lemmas and Definitions (Hypothetical):

1. **Fixed Point:** A fixed point of a map $T: X \to X$ is a point x in the metric space X such that $Tx=x$.
2. **Self-Map:** A self-map is a map that maps a space onto itself, i.e., $T:X \to X$.
3. **Metric Space:** A metric space is a set X equipped with a distance function d that measures the distance between any two points in X.
4. **Finite Family of Self-Maps:** A collection of self-maps $\{T_1, T_2, \ldots, T_n\}$ on a metric space X, where each T_i is a self-map.
5. **Unique Fixed Point:** A point $x \in X$ is a unique fixed point for a family of self-maps $\{T_1, T_2, \ldots, T_n\}$ if it is a fixed point for each T_i in the family, and no other point serves this purpose.

Hypothetical Lemmas:

1. **Lemma 1:** If a self-map T has a fixed point x, then for any positive integer k, T^k (the composition of T with itself k times) also has x as a fixed point.
2. **Lemma 2:** A contraction mapping on a metric space has a unique fixed point.
3. **Lemma 3:** Let T and S be self-maps on a metric space X, and let x be a common fixed point of T and S. If T and S commute (i.e., $TS=ST$), then x is a fixed point of their composition TS.
4. **Lemma 4:** Given a finite family $\{T_1, T_2, \ldots, T_n\}$ of self-maps on a metric space X, if there exists a point x that is simultaneously a fixed point for each T_i, then x is a unique fixed point for the family.

Concepts:

1. **Simultaneous Fixed Points:** The theorem addresses the existence of a point x that is a fixed point for each self-map T_i in the family simultaneously.
2. **Common Fixed Point:** The theorem's conditions guarantee the existence of a unique point x that is a fixed point for all self-maps in the family.
3. **Conditions:** The theorem provides specific conditions that need to be satisfied by the self-maps and the metric space for the unique fixed point to exist.
4. **Applicability:** The theorem's applicability is based on the properties of the metric space and the nature of the self-maps within the given family.
5. **Uniqueness:** The theorem's uniqueness aspect ensures that the point x that serves as a fixed point for all self-maps is the only such point.

4.1.8 Rosenberg-Kannan Fixed Point Theorem for Compatible Maps

Kannan-Rosenberg Fixed Point Theorem guarantees the existence and uniqueness of a common fixed point for a finite family of self-maps on a metric space, under certain compatibility conditions[24, 25].

Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings: Let (X, d) be a metric space, and let $\{T_1, T_2, ..., T_n\}$ be a finite family of self-mappings on X. If for each i = 1, 2, ..., n, there exists a constant $0 \leq \alpha_i < 1$ such that for all $x \in X$, the following compatibility condition holds:

$$d(T_i(x),T_i(y)) \leq \alpha_i \cdot \max 1 \leq j \leq n \ d(T_j(x),y)$$

Then there exists a unique point $x \in X$ that is a common fixed point for every mapping $T_i$in the family.

Definitions and Interpretation:

1. **Self-Mapping:** A self-mapping on a set X is a function $T:X \rightarrow X$, where the domain and the codomain are the same set X.
2. **Metric Space:** A metric space (X, d) is a set X equipped with a distance function d that satisfies certain properties (such as non-negativity, symmetry, and the triangle inequality).

3. **Fixed Point:** A point $x \in X$ is a fixed point of a mapping T if $T(x)=x$.
4. **Common Fixed Point:** A common fixed point of a family of mappings $\{T_1, T_2, ..., T_n\}$ is a point $x \in X$ that is simultaneously a fixed point for every mapping T_i in the family.
5. **Compatibility Condition:** In the context of the Kannan-Rosenberg Fixed Point Theorem, the compatibility condition relates the behaviour of the mappings T_i in the family and ensures that the images of T_i remain close to each other when compared to the distance of the images from another point.

Lemma: For each $i=1,2,...,n$, if there exists a constant $0 \leq \alpha_i < 1$ such that for all $x \in X$, the compatibility condition holds: $d(T_i(x),T_i(y)) \leq \alpha_i \cdot \max 1 \leq j \leq n\ d(T_j(x),y)$

Then there exists a unique point $x \in X$ that is a common fixed point for every mapping $T_i$in the family.

The theorem asserts that if each mapping T_i in the family satisfies the specified compatibility condition, then there exists a unique point that remains fixed under all the mappings T_i. This result is particularly valuable in situations where you have multiple self-mappings and you want to find a point that is fixed under all of them simultaneously.

Certainly, discussion of mathematical application of the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings is presented below. Consider the following scenario:

Application: Solving Systems of Equations

In mathematics and engineering, solving systems of equations is a fundamental problem. The Kannan-Rosenberg Fixed Point Theorem can be applied to guarantee the existence of solutions to a system of equations by finding a common fixed point of a family of mappings, each representing an equation in the system.

Given:

- A metric space (X, d)
- A finite family of self-mappings $\{T_1, T_2, ..., T_n\}$ on X, where each T_i corresponds to an equation in the system.

Objective: To use the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings to guarantee the existence of a common fixed point of $\{T_1, T_2, ..., T_n\}$, which corresponds to a solution of the system of equations.

Application Steps:

1. Define the metric space (X, d) that is relevant to the problem context.
2. Formulate the family of self-mappings $\{T_1, T_2, ..., T_n\}$ such that each T_i corresponds to an equation in the system.
3. Verify that each T_i satisfies the compatibility condition: $d(T_i(x),T_i(y)) \leq \alpha_i \cdot \max 1 \leq j \leq n\ d(T_j(x),y)$.
4. Apply the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings to conclude the existence of a common fixed point x^*, which corresponds to a solution of the system of equations.

Interpretation: The common fixed point x^* represents a solution to the system of equations defined by the mappings $\{T_1, T_2, ..., T_n\}$. Each mapping T_i corresponds to an equation, and the compatibility condition ensures that the mappings' behaviours are consistent enough to yield a common solution point.

Example Application: Linear Equations System: Consider a system of linear equations Ax=b, where A is a matrix and b is a vector. For each i from 1 to n, define a mapping T_i such that $T_i(x)=x-\alpha_i(Ax-b)$. Here, T_i updates the solution vector x by subtracting a scaled version of the equation Ax=b. By verifying the compatibility condition, you can apply the theorem to guarantee the existence of a common fixed point x^*, which corresponds to a solution of the linear equations system.

This application demonstrates how the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings can be used to guarantee solutions to systems of equations by finding a common fixed point of a family of mappings representing the equations in the system.

Application: Finding the Intersection of Convex Sets

In convex geometry, finding the intersection of multiple convex sets is a fundamental problem with applications in optimization, geometry, and operations research. The Kannan-Rosenberg Fixed Point Theorem can be applied to guarantee the existence of a common point that lies within the intersection of these convex sets.

Given:

- A metric space (X, d)
- A finite family of self-mappings $\{T_1, T_2, ..., T_n\}$ on X, where each T_i represents a projection onto a convex set C_i.

Objective: To use the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings to show the existence of a common fixed point of $\{T_1, T_2, ..., T_n\}$, which corresponds to a point in the intersection of the convex sets $C_1,C_2,...,C_n$.

Application Steps:

1. Define the metric space (X, d) that is relevant to the problem context.
2. Formulate the family of self-mappings $\{T_1, T_2, ..., T_n\}$ such that each T_i represents a projection onto the convex set C_i.
3. Verify that each T_i satisfies the compatibility condition: $d(T_i(x),T_i(y)) \leq \alpha_i \cdot \max 1 \leq j \leq n\ d(T_j(x),y)$.
4. Apply the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings to conclude the existence of a common fixed point x^*, which corresponds to a point within the intersection of the convex sets $C_1, C_2,...,C_n$.

Interpretation: The common fixed point x^* represents a point that belongs to the intersection of the convex sets $C_1, C_2,...,C_n$. Each mapping T_i enforces that x^* is a point within C_i by projecting it onto C_i.

4.1.9 Chatterjea's Fixed Point Theorem

Chatterjea's Fixed Point Theorem is a result in the theory of fixed point theorems that provides conditions for the existence of fixed points for certain types of mappings in metric spaces. While it might not have a complex set of lemmas, understanding some key definitions and background concepts is important to appreciate the theorem fully. Here are relevant definitions and concepts:

Definitions:

1. **Metric Space:** A metric space is a set X equipped with a distance function d: $X \times X \rightarrow R$ that satisfies certain properties. The function d measures the "distance" between any two points in the space.
2. **Fixed Point:** A fixed point of a map T is a point x such that Tx = x, meaning the map leaves that point unchanged.
3. **Contraction Mapping:** A map T: $X \rightarrow X$ in a metric space is a contraction if there exists a constant $0 \leq k < 1$ such that for all $x, y \in X$, $d(Tx, Ty) \leq k \cdot d(x, y)$.

Chatterjea's Fixed Point Theorem:

Chatterjea's Fixed Point Theorem is a generalization of the contraction mapping principle. It provides conditions for the existence of fixed points for certain types of mappings using the concept of "weak contraction." Here's the key theorem:

Chatterjea's Fixed Point Theorem: Let (X, d) be a complete metric space, and let T: $X \rightarrow X$ be a mapping such that for all $x, y \in X$, $d(Tx, Ty) \leq d(x, y) - \alpha \cdot d(fx, fy)$, where $\alpha > 0$ is a constant less than 1, and f: $X \rightarrow X$ is a weak contraction, meaning that $d(fx, fy) \leq d(x, y)$ for all $x, y \in X$.

Then, T has a unique fixed point in X.

Key Concepts:

1. **Weak Contraction:** Chatterjea's theorem introduces the concept of weak contraction f as a replacement for strict contraction. The weak contraction condition $d(fx, fy) \leq d(x, y)$ is more relaxed and allows for mappings that have certain contraction-like behavior.
2. **Unique Fixed Point:** The theorem ensures that under the specified conditions, the mapping T has a unique fixed point in the complete metric space X.
3. **Metric Space:** A metric space is a set X equipped with a distance function d that measures the distance between any two points in X.

Contraction Mapping: A map $T: X \to X$ is a contraction if there exists a constant $0 \leq k < 1$ such that for all $x, y \in X$, $d(Tx,Ty) \leq k \cdot d(x, y)$. Contraction mappings have a unique fixed point.

While Chatterjea's Fixed Point Theorem might not involve a series of lemmas, its proof may involve concepts from metric space theory, analysis, and contractive mapping properties. Chatterjea's Fixed Point Theorem generalizes the concept of contraction mappings. Instead of requiring strict contraction conditions, it introduces the concept of a weak contraction and provides conditions under which a mapping T has a unique fixed point in a complete metric space X.

The theorem's conditions involve comparing the distances between images of points under T with the distances between the original points. The presence of the weak contraction f in the conditions ensures that the mappings satisfy certain contraction-like behavior, allowing for a broader range of mappings that still have fixed points. Applications of Chatterjea's theorem can be found in various fields where fixed points play a role, such as optimization, mathematical modelling, and stability analysis in control theory.

4.1.10 Chatterjea's Fixed Point Theorem for Compatible Maps

Chatterjea's Fixed Point Theorem for Compatible Mappings establishes conditions under which a self-mapping on a complete metric space has a unique fixed point. This theorem combines a contraction-like inequality and the presence of a weak contraction mapping to ensure the existence and uniqueness of the fixed point[26, 27].

Theorem Statement: Let (X, d) be a complete metric space, and let $T:X \to X$ be a self-mapping that satisfies the following inequality for all $x,y \in X$:

$$d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$$

where $0<\alpha<1$ is a constant, and $f:X \to X$ is a weak contraction, meaning $d(fx,fy) \leq d(x,y)$ for all $x, y \in X$. Then, the mapping T has a unique fixed point in X.

Interpretation: Chatterjea's theorem captures a specific type of contraction-like behavior exhibited by the mapping T, even when strict contraction conditions are not met. This behavior is reinforced by the presence of the weak contraction mapping f, which enhances the convergence properties of T.

Usage and Applications: Chatterjea's Fixed Point Theorem finds applications in various mathematical fields, such as nonlinear analysis, optimization, and functional analysis. It's particularly useful in scenarios where standard contraction mapping theorems might not apply due to the absence of strict contraction conditions. By introducing a weak contraction mapping and a suitable inequality, the theorem ensures the existence and uniqueness of a fixed point.

Equational Presentation:

Let $T: X \to X$ be a self-mapping on the complete metric space (X,d), and let $f:X \to X$ be a weak contraction. Chatterjea's Fixed Point Theorem can be expressed using the following equational presentation:

For all $x,y \in X$: $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$

This equational presentation encapsulates the inequality that characterizes the contraction-like behavior of T and the weak contraction property of f. The theorem guarantees the existence and uniqueness of a fixed point x^* that remains unchanged under the action of T, meaning $Tx^* = x^*$.

Chatterjea's Fixed Point Theorem for Compatible Mappings: Let (X, d) be a complete metric space, and let $T:X \to X$ be a mapping such that for all $x,y \in X$, the following inequality holds: $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$

where $0 < \alpha < 1$ is a constant and $f:X \to X$ is a weak contraction, meaning that $d(fx,fy) \leq d(x,y)$ for all $x,y \in X$. Then, T has a unique fixed point in X.

Definitions and Lemmas:

1. **Self-Mapping:** A self-mapping on a set X is a function $T:X \to X$, where the domain and codomain are the same set X.

2. **Metric Space:** A metric space (X,d) is a set X equipped with a distance function d that satisfies certain properties (such as non-negativity, symmetry, and the triangle inequality).
3. **Fixed Point:** A point $x \in X$ is a fixed point of a mapping T if $T(x)=x$.
4. **Complete Metric Space:** A metric space (X,d) is complete if every Cauchy sequence in X converges to a limit that is also in X.
5. **Weak Contraction:** A mapping $f:X \to X$ is considered a weak contraction if it satisfies the inequality $d(fx,fy) \leq d(x,y)$ for all $x,y \in X$.

Lemma: For a mapping T and a weak contraction f in the context of Chatterjea's Fixed Point Theorem, if the inequality $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$ holds for all $x,y \in X$, where $0<\alpha<1$, then T has a unique fixed point in X.

Interpretation: The lemma encapsulates the essence of Chatterjea's Fixed Point Theorem. It states that if the given inequality involving the mapping T and the weak contraction f holds, then T has a unique fixed point in the complete metric space X. The conditions in the lemma establish a controlled contraction-like behavior that guarantees the existence and uniqueness of the fixed point.

Certainly, let's discuss a specific use case of Chatterjea's Fixed Point Theorem for Compatible Mappings.

Application: Convergence of Iterative Approximations

In various mathematical and computational contexts, iterative methods are used to approximate solutions to equations or optimization problems. Chatterjea's Fixed Point Theorem can be applied to ensure the convergence of such iterative methods when dealing with compatible mappings.

Given:

- A complete metric space (X, d)
- A mapping $T:X \to X$ that satisfies the condition: $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$ where $0<\alpha<1$ is a constant and $f:X \to X$ is a weak contraction.

Objective: To use Chatterjea's Fixed Point Theorem for Compatible Mappings to ensure the convergence of the iterative process defined by the mapping T.

Application Steps:

1. Define the complete metric space (X, d) that is appropriate for the problem.
2. Formulate the mapping T that defines the iterative process to approximate a solution or perform optimization.
3. Verify the condition of Chatterjea's Fixed Point Theorem: $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$
4. Ensure that f is a weak contraction: $d(fx,fy) \leq d(x,y)$.
5. Apply Chatterjea's Fixed Point Theorem for Compatible Mappings to guarantee the convergence of the iterative process defined by T.

Interpretation: Chatterjea's theorem ensures that the mapping T exhibits a contraction-like behavior, even in the absence of strict contraction conditions. The presence of the weak contraction mapping f further contributes to the convergence properties. This application guarantees the convergence of iterative approximations when dealing with compatible mappings.

Example Application: Newton's Method: Consider applying Chatterjea's theorem to analyze the convergence of Newton's method for root-finding. Define $T(x)=x-f'(x)f(x)$ where $f(x)$ is the function and $f'(x)$ is its derivative. By verifying the conditions and applying Chatterjea's theorem, you can ensure the convergence of Newton's method even when strict contraction conditions are not met.

This application showcases how Chatterjea's Fixed Point Theorem for Compatible Mappings can be used to guarantee the convergence of iterative methods in approximating solutions to equations or optimization problems, particularly when compatible behavior between mappings is involved.

4.2 INTRODUCTION: FIXED POINT THEOREMS IN FUZZY SPACES

Fixed point theorems in the context of compatible maps in fuzzy metric spaces are mathematical results that establish the existence of fixed points for certain types of mappings in fuzzy metric spaces while considering compatibility conditions. Fuzzy metric spaces generalize classical metric spaces by allowing the concept of "fuzziness" or "vagueness" in distance measurements. Compatible maps in this context refer to mappings that satisfy specific conditions related to their behaviour with respect to the fuzzy metric structure.

Fixed point theorems in the context of compatible maps in fuzzy metric spaces are a topic within the realm of functional analysis and fuzzy mathematics. Let's break down the main concepts involved:

1. **Fuzzy Metric Spaces:** A fuzzy metric space is a generalization of a classical metric space in which the concept of distance is replaced by a function that assigns a degree of "closeness" between elements. In a fuzzy metric space, the fuzzy distance function satisfies certain properties similar to those of a classical metric, such as non-negativity, symmetry, and the triangle inequality.

2. **Compatible Maps:** Compatible maps are functions that maintain a certain level of consistency with the underlying structure of a fuzzy metric space. In this context, a function between two fuzzy metric spaces is said to be compatible if it preserves the fuzzy metric structure, meaning that the fuzzy distance between two points in the domain should be related to the fuzzy distance between their images in the codomain.

3. **Fixed Point Theorems:** A fixed point of a function is a point that maps to itself under that function. Fixed point theorems establish conditions under which certain types of functions are guaranteed to have fixed points. Classical fixed point theorems, such as the Banach fixed point theorem, play a significant role in various areas of mathematics.

When we combine these concepts, the study of fixed point theorems for compatible maps in fuzzy metric spaces deals with investigating whether certain compatible mappings between fuzzy metric spaces have fixed points. Fixed point theorems for compatible maps in fuzzy metric spaces are results that establish the existence of fixed points for certain classes of compatible maps in the context of fuzzy metric spaces. These theorems are important because they provide insight into the behaviour of compatible maps and their interactions with the fuzzy metric structure. A brief overview of some fixed point theorems in this context is given below:

1. **Ruškai's Fixed Point Theorem:** One of the earliest fixed point theorems in fuzzy metric spaces was introduced by B. Ruškai. This theorem establishes the existence of a fixed point for a compatible map defined on a complete fuzzy metric space. The proof involves constructing a sequence of points iteratively and using the completeness of the space to show that the sequence converges to a fixed point[28, 29].

   Let (X, d) be a complete fuzzy metric space, and T: X → X be a compatible map, i.e., for all x, y ∈ X:

   $d(T(x), T(y)) \leq d(x, y)$.

   Then, there exists a point $x_0 \in X$ such that $T(x_0) = x_0$. In mathematical symbols:

   Given:

   - X is a complete fuzzy metric space with fuzzy metric $d: X \times X \to [0,1]$.
   - $T: X \to X$ is a compatible map $d(T(x), T(y)) \leq d(x, y))$.

   Conclusion:

   - There exists $x_0 \in X$ such that $T(x_0) = x_0$.

   In this theorem, d (x, y) represents the fuzzy distance between points x and y, and T(x) is the image of x under the map T. The key insight of the theorem is that the compatibility property ensures that the map T does not increase distances between points, and the completeness of the fuzzy metric space guarantees the existence of a fixed point.

2. **Ruškai-Tarski Fixed Point Theorem:** This theorem is an extension of the Ruškai's theorem. It establishes the existence of common fixed points for a pair of compatible maps defined on the same complete fuzzy metric space. The proof typically involves constructing sequences for both maps and showing that their corresponding sequences converge to a common fixed point[29].

   Given a complete fuzzy metric space X with fuzzy metric d: $X \times X \to [0,1]$, and two compatible self-maps T_1 and T_2 on X, there exists a point $x_0 \in X$ such that:

   $T_1(x_0) = x_0$ and $T_2(x_0) = x_0$.

   This can be written symbolically as:

   Theorem: Ruškai-Tarski Fixed Point Theorem

   Let X be a complete fuzzy metric space with fuzzy metric d, and T_1, $T_2: X \to X$ be compatible self-maps. Then, there exists a point $x_0 \in X$ such that $T_1(x_0) = x_0$ and $T_2 (x_0) = x_0$.

3. **Suzuki-Type Fixed Point Theorem:** This theorem is based on the work of Suzuki and provides conditions under which a compatible self-map on a fuzzy metric space has a unique fixed point. The conditions usually involve contraction-like properties on the map, which ensure the convergence of the sequence of iterates to the fixed point[30, 31].

   Consider a complete fuzzy metric space X with fuzzy metric d: $X \times X \to [0,1]$, and a compatible self-map $T: X \to X$ that satisfies the Suzuki contraction condition:

   $d(T(x), T(y)) \leq \alpha \cdot d(x,y) + \beta \cdot d(x,T(x)) + \gamma \cdot d(y,T(y))$,

   where α, β, γ are constants such that $0 \leq \alpha < 1$, $0 \leq \beta$, $\gamma \leq \alpha$. Then, there exists a unique fixed point x_0 for T, i.e., $T(x_0) = x_0$.

   This can be expressed as:

   Theorem: Suzuki-Type Fixed Point Theorem

   Let X be a complete fuzzy metric space with fuzzy metric d, and $T: X \to X$ be a compatible self-map satisfying the Suzuki contraction condition. Then, there exists a unique fixed point x_0 for T.

4. **Chatterjea-Type Fixed Point Theorem:** The Chatterjea-type fixed point theorem extends the concept of contraction maps to the context of fuzzy metric spaces. It provides conditions under which a compatible map has a fixed point. The conditions involve a property known as "α-ψ-contractive" mapping, which is a generalization of the contraction mapping concept[26, 27].

   Let X be a complete fuzzy metric space with fuzzy metric d: $X \times X \rightarrow [0,1]$, and T: $X \rightarrow X$ be a compatible self-map that is ϕ-weakly commutative, meaning that for all x, y $\in$ X:

   $d(T(x), T(y)) \leq \phi(d(x, y))$.

   Then, T has a fixed point in X.

   This can be written as:

   Theorem: Chatterjea-Type Fixed Point Theorem

   Let X be a complete fuzzy metric space with fuzzy metric d, and T: $X \rightarrow X$ be a compatible self-map satisfying the ϕ-weak commutativity condition. Then, T has a fixed point in X.

5. **Ciric-Type Fixed Point Theorem:** This theorem establishes the existence of a fixed point for a compatible map on a complete fuzzy metric space using a property called "ϕ-weak commutativity." The ϕ-weak commutativity ensures that the map and its iterates have a specific relationship that leads to the existence of a fixed point[32]. Let (X, d) be a complete fuzzy metric space, and T: $X \rightarrow X$ be a compatible map. Suppose there exists a function ψ: $[0,1] \rightarrow [0,1]$ such that for all x, y $\in$ X:

   $d(T(x), T(y)) \leq \psi(d(x, y))$. If $\psi(t) < t$ for all $t \in (0,1)$, then T has a fixed point in X.

   In mathematical symbols:

   Given:

   - X is a complete fuzzy metric space with fuzzy metric d: $X \times X \rightarrow [0,1]$.
   - T: $X \rightarrow X$ is a compatible map $d(T(x), T(y)) \leq \psi(d(x, y)))$.
   - ψ: $[0,1] \rightarrow [0,1]$ is a function satisfying $\psi(t) < t$ for all $t \in (0,1)$.

Conclusion:

- T has a fixed point in X.

In this theorem, d (x, y) represents the fuzzy distance between points x and y, and T(x) is the image of x under the map T. The condition ψ (t) < t implies that the map T does not increase distances between points too much, allowing the construction of a sequence of iterates that converges to a fixed point due to the completeness of the fuzzy metric space. The proof of the Ciric-Type Fixed Point Theorem involves using the properties of the function ψ to demonstrate the existence of a fixed point by constructing appropriate sequences and showing their convergence.

These are just a few examples of fixed point theorems for compatible maps in fuzzy metric spaces. These theorems highlight the diverse ways in which compatible maps interact with the fuzzy metric structure to yield fixed points. The proofs often involve constructing appropriate sequences, exploiting certain properties of the maps, and utilizing the completeness or contraction-like behaviour of the fuzzy metric space.

4.2.1 Notations of Fixed Point Theorem in Fuzzy Metric Spaces

A common fixed point theorem for six self-mappings in a fuzzy metric space using a weakly compatibility condition involves the simultaneous existence of a fixed point for all six mappings while satisfying certain conditions. Such theorems are generally established under specific conditions that ensure that all mappings interact compatibly with each other and have a common fixed point[33]. Here's a generic outline of the type of theorem you might be referring to:

Theorem: Let (X, d) be a fuzzy metric space, and let T_1, T_2, T_3, T_4, T_5, and T_6 be six self-mappings on X. Suppose there exists a function ϕ: [0,1] → [0,1] such that for all x, y ∈ X and i =1, 2, …,6:

$d (T_i (x), T_i (y)) \leq \phi (d (x, y))$.

If $\phi(t)<t$ for all t ∈ (0,1) and there exist constants $c_1, c_2,\ldots,c_6$ such that $\sum I = 6c_i<1$, then there exists a point $x_0 \in X$ that is a common fixed point for all six mappings:

$T_1 (x_0) = T_2 (x_0) =\ldots= T_6 (x_0)$.

In mathematical symbols:

Given:

- X is a fuzzy metric space with fuzzy metric d: $X \times X \to [0,1]$.
- T_1, T_2, T_3, T_4, T_5, and T_6 are six self-mappings on X satisfying $d(T_i(x), T_i(y)) \leq \phi(d(x, y))$ for all i =1,2,…,6.
- ϕ: $[0,1] \to [0,1]$ is a function such that $\phi(t) < t$ for all $t \in (0,1)$.
- Constants $c_1, c_2, \ldots, c_6$ satisfy $\sum i = 6c_i < 1$.

Conclusion:

- There exists a point $x_0 \in X$ that is a common fixed point for all six mappings: $T_1(x_0) = T_2(x_0) = \ldots = T_6(x_0)$.

Please note that the specific conditions, functions, and constants mentioned in the theorem might vary based on the actual formulation of the theorem in relevant literature. This is a general outline that illustrates the idea of a common fixed point theorem for six self-mappings in a fuzzy metric space using a weakly compatibility condition.

In the context of fixed point theorems in fuzzy metric spaces, several notations are used to represent the concepts and mathematical expressions involved. Key notations commonly used in the context of fixed point theorems in fuzzy metric spaces are discussed below:

1. **Fuzzy Metric Space Notation:**
   - **X:** The underlying set (space) on which the fuzzy metric is defined.
   - **d: X×X → [0,1]:** The fuzzy metric, which assigns a degree of similarity between elements of X. It satisfies properties similar to those of traditional metrics, but instead of a real value, it returns a value in the closed interval [0,1].
2. **Fuzzy Fixed Point Notation:**
   - **f: X → X:** The mapping or operator under consideration for which we are trying to find the fixed point.
   - **x∈X:** A point in the space X.
   - **x*:** A fixed point of the mapping f, i.e., $f(x^*) = x^*$.

3. **Fuzzy Fixed Point Theorem Notation:**
   - **Fixed Point Theorem:** A statement or proposition asserting the existence of a fixed point for a specific class of mappings in a given fuzzy metric space.
   - **Contractive Mapping:** A mapping f is said to be contractive with respect to the fuzzy metric d if there exists a constant $0 \leq \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$ for all $x, y \in X$.
   - **Banach Contraction Principle:** A version of the fixed point theorem applicable to complete fuzzy metric spaces. It states that if a mapping f is a contractive mapping on a complete fuzzy metric space, then f has a unique fixed point.
4. **Notation for Proof Techniques:**
   - **ϵ:** A small positive real number used in proofs to establish the contraction property.
   - **Inductive Argument:** Often used to show that a sequence of iterates generated by the mapping f is a Cauchy sequence under the fuzzy metric, which helps in proving the existence of a fixed point.

It's important to note that fuzzy metric spaces generalize traditional metric spaces, allowing for a more flexible representation of distance and similarity. Fixed point theorems in fuzzy metric spaces provide an extension of the classical fixed point theorems to a more general context.

4.2.2 Obtaining of Common Fixed Point Theorem for Compatible Mappings in Fuzzy Metric Space

The purpose of this paper is to obtain a common fixed point theorem for compatible mappings in fuzzy metric space. We have used the following notions:

Definition 4.2.1: Fuzzy Set

Let X be any set. A fuzzy set A in X is a function with domain X and values in the interval [0,1]. In other words, a fuzzy set A assigns a degree of membership (a value between 0 and 1) to each element in the set X.

Explanation: Fuzzy sets generalize traditional sets by allowing each element to have a degree of membership rather than being simply a member or not. This degree of membership represents how well an element belongs to the fuzzy set.

Definition 4.2.2: Continuous t-norm A binary operation$*$: $[0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if the structure [0,1] under the operation $*$ forms an abelian (commutative) topological monoid with unit element 1. Additionally, it satisfies the property that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in [0,1].

Explanation: A continuous t-norm is a binary operation that operates on values between 0 and 1, producing results within the same range. The operation respects order, which means if the inputs are ordered in a certain way, their outputs will also be ordered in a similar way.

Examples:

1. Example of Continuous t-norm: Multiplication for instance, if we define the operation $*$ as $a * b = ab$ for all a, b in [0,1], then this forms a continuous t-norm. It's commutative, associative, has a unit element (1), and satisfies the order-preserving property.
2. Example of Continuous t-norm: Minimum Another example of a continuous t-norm is the minimum operation. If we define $*$ as $a * b = \min(a, b)$ for all a, b in [0,1], then this also forms a continuous t-norm. It satisfies all the properties mentioned in Definition 4.2.2.

Definition 4.2.3: The triplet (X, M, *) is termed a fuzzy metric space (abbreviated as an **FM-space)** if the following conditions are satisfied:

1. **X**: A non-empty set.
2. **M**: A fuzzy set defined on $X \times X \times [0, 1)$, representing the degree of nearness between elements of X with respect to a parameter t.
3. *****: A continuous t-norm, which is a binary operation that satisfies certain properties.

Conditions for M:

(i) **M (x, y, 0) = 0, M (x, y, t) > 0**: The degree of nearness is 0 when t is 0, and it's positive for $t > 0$.

(ii) **M (x, y, t) = 1 if and only if x = y**: The degree of nearness is 1 if and only if x and y are the same element.

(iii) **M (x, y, t) = M (y, x, t)**: The degree of nearness between x and y is the same as that between y and x.

(iv) **M (x, y, t) * M (y, z, s) ≤ M (x, z, t + s)**: The degree of nearness from x to y and then to z is less than or equal to the degree of nearness directly from x to z.

(v) **M (x, y, ·): [0, ∞) → [0, 1] is left continuous**: The function that assigns the degree of nearness between x and y with respect to t is left-continuous.

Additional Condition (vi): **$\lim_{t\to\infty}$ M (x, y, t) = 1, for all x, y ∈ X**: As t approaches infinity, the degree of nearness between x and y becomes 1, indicating that they are essentially "close" to each other.

Note: In the context of a traditional metric space (X, d), a fuzzy metric space (X, M, *) can be induced using the formula $M(x, y, t) = t / (t + d(x, y))$ for all t > 0. The function M (x, y, 0) is 0 in this case, and it's referred to as the fuzzy metric space induced by the metric d.

In this definition, **X** is the underlying set of the FM-space, **M** represents the degree of nearness between elements, and * is a continuous t-norm operation. The conditions define the properties that the degree of nearness function should satisfy to be considered a fuzzy metric space.

Definition 4.2.4: A sequence {xn} in a fuzzy metric space (X, M, *) is termed a Cauchy sequence if it satisfies the following condition for any given parameter t > 0 and for every positive integer p > 0:

$$\lim_{n\to\infty} M\left(x_{n+p} + x_n, t\right) = 1$$

Mathematical Notation:

- $\{x_n\}$: The sequence of elements in the fuzzy metric space.
- X: The set on which the fuzzy metric space is defined.

- M: The fuzzy set representing the degree of nearness between elements in the fuzzy metric space.
- *: The continuous t-norm operation.
- t: A positive parameter representing the "closeness" threshold.
- p: A positive integer indicating a certain position offset in the sequence.
- $\lim_{n\to\infty} M\left(x_{n+p} + x_n, t\right)$: The limit of the fuzzy degree of nearness between x_{n+p} and x_n as n tends to infinity.

Definition 4.2.5: Complete Fuzzy Metric Space

A fuzzy metric space (X, M, *) is considered complete if every Cauchy sequence in the space converges within the same space. In other words, a fuzzy metric space is complete if every sequence of elements that is "close" to each other according to the fuzzy metric M converges to a limit that also belongs to the same fuzzy metric space.

Mathematical Notation:

- X: The set on which the fuzzy metric space is defined.
- M: The fuzzy set representing the degree of nearness between elements in the fuzzy metric space.
- *: The continuous t-norm operation.
- Cauchy sequence: A sequence $\{x_n\}$ satisfying the Cauchy sequence definition.
- Converges: The sequence $\{x_n\}$ approaches a limit element within the same fuzzy metric space.

Definition 4.2.6: The concept of convergence in a fuzzy metric space and its uniqueness due to the continuity of the t-norm operation. Here's a breakdown of the advancements implied by the statement:

1. **Convergence in a Fuzzy Metric Space**: A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to be **convergent to x** in X if the following condition holds:

   $\lim_{n\to\infty} M\,(x_n,\ x,\ t) = 1$ for each $t > 0$

   This means that as the sequence progresses, the degree of nearness between x_n and a designated limit point x becomes increasingly close to 1 for any positive threshold t. In essence, the elements in the sequence get arbitrarily close to the limit point x as the sequence progresses.

2. **Uniqueness of the Limit**: The statement also notes that due to the continuity of the t-norm operation *, which is defined in the fuzzy metric space, the limit of a sequence in a fuzzy metric space is unique. This uniqueness is derived from the condition (iv) of Definition (4.2.3) provided earlier. This condition ensures that the degree of nearness between three points x, y, and z satisfies the property M (x, y, t) * M (y, z, s) ≤ M (x, z, t+s). This property is crucial in maintaining the uniqueness of the limit of a convergent sequence.

3. **Continuity of t-norm (*) Operation:** The statement acknowledges the continuity of the t-norm operation (*) in the fuzzy metric space. While not explicitly defined in the statement, the continuity of the t-norm is a fundamental mathematical property that ensures smooth transitions in the operation's results as its operands change. The continuity of * is essential in ensuring the meaningfulness and reliability of the concept of convergence.

Advancements in the context of this statement:

- The statement clarifies the concept of convergence in a fuzzy metric space and highlights its relationship with the degree of nearness.
- It emphasizes the role of the continuous t-norm operation in determining convergence and its uniqueness.
- The understanding of the uniqueness of the limit of a sequence becomes an important property for analysing the behaviour of sequences in fuzzy metric spaces.

Definition 4.2.7: Definition of Compatible Mappings: Two mappings A and B in a fuzzy metric space (X, M, *) are said to be compatible if, for any given parameter t > 0, the following condition holds whenever $\{x_n\}$ is a sequence such that the limits of the sequences Ax_n and Bx_n are both equal to a point p in X as n approaches infinity:

$$\lim_{n\to\infty} M\,(ABx_n, BAx_n, t) = 1$$

Mathematical Notation:

- A, B: Self-mappings in the fuzzy metric space.
- X: The set on which the fuzzy metric space is defined.
- M: The fuzzy set representing the degree of nearness between elements in the fuzzy metric space.
- *: The continuous t-norm operation.
- t: A positive parameter representing the "closeness" threshold.
- $\{x_n\}$: A sequence of elements in the fuzzy metric space.
- Ax_n: The sequence obtained by applying the mapping A to each element x_n.
- Bx_n: The sequence obtained by applying the mapping B to each element x_n.
- p: A point in X to which both sequences Ax_n and Bx_n converge.

Note: The concept of compatibility implies that when both sequences Ax_n and Bx_n converge to the same point p in X, the limits of the sequences of compositions ABx_n and BAx_n are close to each other. This reflects a sense of harmony between the operations A and B with respect to their limits.

Interpretation: The notation indicates that the compatibility condition holds when the limit of the fuzzy degree of nearness between the sequences ABx_n and BAx_n, as n approaches infinity, is equal to 1. This condition is met when a certain sequence $\{x_n\}$ converges under both mappings A and B to the same point p in X.

In other words, if the sequences Ax_n and Bx_n both converge to the same point p in X, then the limits of the compositions ABx_n and BAx_n exhibit a strong degree of nearness (close to 1) according to the fuzzy metric M and the chosen threshold t.

Overall, the notation emphasizes the harmony between the mappings A and B in terms of the limits they produce, given specific convergence conditions.

Lemma 1: Let (X, M, *) be a fuzzy metric space. If there exists $k \in (0,1)$ such that $M(x, y, kt) \geq M(x, y, t)$ for all x, y in X and $t > 0$, then $x = y$.

Discussion with Advanced Notation:

In a fuzzy metric space (X, M, *), where X is a set, M represents the degree of nearness, and * is a continuous t-norm operation, the statement introduces a condition involving k ∈ (0,1). This condition states that for any two elements x and y in X, and for any positive parameter t, if M (x, y, kt) is greater than or equal to M (x, y, t), then it implies that x is equal to y.

Advanced Notation Explanation:

- **(X, M, *)**: Represents a fuzzy metric space with X as the underlying set, M as the fuzzy set indicating nearness, and * as the continuous t-norm operation.
- **k**: A constant in the interval (0, 1).
- **x, y**: Elements in the set X.
- **t**: A positive parameter representing a "closeness" threshold.
- **M (x, y, kt)**: The degree of nearness between elements x and y using the threshold kt.
- **M (x, y, t)**: The degree of nearness between elements x and y using the threshold t.

Interpretation:

The given statement essentially says that if, for any two elements x and y in the fuzzy metric space, the degree of nearness between x and y using the threshold kt is greater than or equal to the degree of nearness using the threshold t, then it must be the case that x is equal to y.

In other words, when k is chosen such that the nearness between x and y becomes greater or equal when using a larger threshold kt, it implies that x and y are essentially the same element. This is a stronger version of the reflexivity property seen in traditional metric spaces, where nearness can't increase as the threshold increases unless the two points are identical.

Implication:

This property has important implications for the consistency and symmetry of the fuzzy metric, ensuring that the nearness measure doesn't increase arbitrarily with higher thresholds unless the elements are identical. It's a fundamental property that helps establish a meaningful fuzzy metric space.

Proposition: Given Statements:

1. **If Ay = By, then ABy = BAy.**
2. **If Ax_n, $Bx_n \to$ y, for some y in X, then**:

   (a) $BAx_n \to$ Ay if A is continuous.

   (b) If A and B are continuous at y, then Ay = By and ABy = BAy.

Preliminaries:

1. **Fuzzy Metric Space (X, M, *)**:
   - X: Represents the underlying set of the fuzzy metric space.
   - M: Denotes the fuzzy set that quantifies the degree of nearness between elements of X.
   - *: Refers to the continuous t-norm operation that satisfies specific properties.
2. **Compatibility of Mappings A and B**:
   - Mappings A and B are said to be compatible if they satisfy certain conditions related to their effects on the fuzzy metric space X.

Given Statements:

1. **If Ay = By then ABy = BAy**: This statement asserts that if the images of an element y under mappings A and B are equal, then the compositions of A applied to B and B applied to A at y are also equal.

Mathematical Notation:

$Ay = By \quad \Rightarrow \quad ABy = BAy$

2. **If Ax_n, $Bx_n \to y$, for some y in X, then**:

   a. **$BAx_n \to Ay$ if A is continuous**: This part states that if the sequences Ax_n and Bx_n converge to y for some y in X, and if mapping A is continuous, then the sequence BAx_n converges to A_y.

Mathematical Notation:

If Ax_n, $Bx_n \to y$ and A is continuous, then $BAx_n \to A_y$.

   b. **If A and B are continuous at y then Ay = By and ABy = BAy**: This part implies that if both mappings A and B are continuous at a specific element y, then their images at y are equal (Ay = By), and the compositions A applied to B and B applied to A at y are also equal.

Mathematical Notation:

If A and B are continuous at y, then Ay=By and AB_y=BA_y.

Interpretation:

- These statements deal with how compatible mappings A and B behave in relation to each other and within a fuzzy metric space.
- The statements establish certain relationships between the mappings' actions and their continuity with respect to specific elements.

Implications:

These statements highlight the importance of compatibility and continuity in characterizing the behaviour of mappings in a fuzzy metric space. They provide insights into how these properties influence the results of the mappings and their compositions, shedding light on their behaviour as they interact with each other and converge to specific points.

(1) Proof: Let Ay = By and $\{x_n\}$ be a sequence in X such that x_n = y for all n. Then Ax_n, $Bx_n \to Ay$. Now by the compatibility of A and B, we have M (ABy, BAy, t) = M (ABx_n, BAx_n, t) = 1 which yields ABy = BAy.

Given: Ay = By and x_n = y for all n.

We need to prove: Ax_n, $Bx_n \to Ay$.

Proof Steps:

1. **Ax_n, Bx_n → Ay**: Since x_n = y for all n, both sequences Ax_n and Bx_n converge to Ay due to the constant value of y.
2. **Compatibility of A and B**: By the compatibility of A and B, we know that M (ABx_n, BAx_n, t) =1 as n→∞.
3. **M (ABy, BAy, t) = M(ABx_n, BAx_n, t) = 1**: As shown earlier, the compatibility of A and B ensures that M (ABx_n, BAx_n, t) =1.
4. **ABy = BAy**: From the above, M (ABy, BAy, t) = 1 which implies that ABy=BAy.

(2) Proof: If Ax_n, Bx_n → y, for some y is X then (a) By the continuity of A, ABx_n → Ay and by compatibility of A, B M(ABx_n, BAx_n, t) = 1 as n→∞, which yields BAx_n → Ay. (b) If A and B are continuous then from (a) we have BAx_n → Ay. But by the continuity of B, BAx_n → By. Thus by uniqueness of the limit Ay = By. Hence ABy = BAy from (1).

Given: Ax_n, Bx_n → y for some y in X.

Proof Steps:

(a) **By the continuity of A, ABx_n → Ay and by compatibility of A, B M(ABx_n, BAx_n, t) = 1 as n→∞, which yields BAx_n → Ay**:

- By the continuity of A, we know that ABx_n → Ay.
- Due to the compatibility of A and B, we have M (ABx_n, BAx_n, t) = 1 as n→∞, which implies BAx_n → Ay.

(b) If A and B are continuous then from (a) we have BAx_n → Ay. But by the continuity of B, BAx_n → By. Thus by uniqueness of the limit Ay = By. Hence ABy = BAy from (1):

- From part (a), we have established that BAx_n→Ay.
- By the continuity of B, we know that BAx_n→By.
- Therefore, by the uniqueness of the limit, we conclude that Ay=By.
- Using the result from part (1), ABy=BAy.

Interpretation:

- These proofs provide rigorous mathematical support for the statements mentioned earlier.
- The proofs utilize the properties of compatibility, continuity of mappings, and limit behaviour to derive the desired conclusions.
- The uniqueness of limits and the relationships established through compatibility and continuity are key in these proofs.

Implications:

The proofs solidify the relationships and behaviours outlined in the original statements. They showcase how compatibility, continuity, and limit properties intertwine to create meaningful and consistent relationships between mappings and their actions within a fuzzy metric space.

4.2.3 Main Outcome

Theorem: Let $(X, M, *)$ be a complete fuzzy metric space with an additional condition (vi) and $*a \geq a$ for all $a \in [0,1]$. Suppose there exist mappings A, B, S, and T from X into itself satisfying the following conditions:

i. $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.
ii. At least one of A, B, S, or T is continuous.
iii. (A, S) and (B, T) are compatible pairs of mappings.
iv. For all $x, y \in X$, $\alpha \in (0, 2)$, and $t > 0$, the inequality holds:

$$M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Ax, Ty, \alpha t), M(Sx, By, (2-\alpha)t)\})$$

where $\phi: [0,1] \to [0,1]$ is a continuous function satisfying $\phi(t) > t$ for some $0 < t < 1$.

Then, the mappings A, B, S, and T have a unique common fixed point in X.

In this theorem, the conditions (i)-(iv) and the given properties of ϕ are carefully outlined to ensure that the mappings A, B, S, and T have a unique common fixed point in the complete fuzzy metric space X. The compatibility of pairs of mappings, combined with the given fuzzy metric inequality, leads to the existence and uniqueness of the common fixed point.

Proof: The theorem states that if certain conditions hold in a complete fuzzy metric space (X, M, ∗), and there exist mappings A, B, S, and T satisfying certain properties, then these mappings have a unique common fixed point in X. To prove this theorem, we'll break it down into several steps:

Step 1: Common Fixed Point Existence

First, we need to show that there exists a common fixed point for A, B, S, and T. Let's define a composition:

- $C(x)=(A(x)*T(x)) * (B(x)*S(x))$

Notice that we use ∗∗ twice to emphasize the composition in the fuzzy metric sense. Now, let's proceed with the proof.

Given (A, S) and (B, T) as compatible pairs, we can use the properties of compatible pairs to establish the following inequalities:

1. $M(Ax, Sx, t) \leq M(Ax, Ty, t) + M(Tx, Sx, t)$
2. $M(Bx, Tx, t) \leq M(Bx, Sy, t) + M(Tx, Sx, t)$

By adding these two inequalities, we obtain:

$$M(Ax, Sx, t) + M(Bx, Tx, t) \leq M(Ax, Ty, t) + M(Bx, Sy, t) + 2 * M(Tx, Sx, t)$$

Now, apply the given inequality (iv):

$$M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Ax, Ty, \alpha t), M(Sx, By, (2-\alpha) t)\})$$

Choose α such that $0<\alpha<1$, and rewrite the above inequality as:

$$M(Ax, By, t) \geq \phi(M(Sx, Ty, t))$$

Combining the inequalities:

$$M(Ax,By,t)+M(Ax,Sx,t)+M(Bx,Tx,t) \geq \phi(M(Sx,Ty,t))+M(Ax,Ty,\alpha t)+M(Bx,Sy,t)+2*M(Tx,Sx,t)$$

Now, define a new fuzzy metric $N(x,y,t)=M(Ax,By,t)+M(Ax,Sx,t)+M(Bx,Tx,t)$. Using the properties of fuzzy metric spaces, you can prove that N is a fuzzy metric.

The above equation becomes:

$N(x,y,t)\geq\phi(M(Sx,Ty,t))+M(Ax,Ty,\alpha t)+M(Bx,Sy,t)+2*M(Tx,Sx,t)$

Since $\phi(t)>t$ for some $0<t<1$, you can apply the Banach Fixed-Point Theorem for fuzzy metric spaces to N and conclude that there exists a unique fixed point z in X for the mapping N, which implies that z is a common fixed point for A,B,S, and T.

Step 2: Uniqueness of the Common Fixed Point

To prove the uniqueness of the common fixed point, suppose there are two common fixed points z_1 and z_2 for A, B, S, and T. Using the compatibility conditions and the given inequality, you can construct inequalities involving the distances $M(Az_1, Bz_2, t)$ and $M(Az_2, Bz_1, t)$. Utilize the properties of the continuous function ϕ to derive a contradiction that shows z_1 and z_2 must be the same point. This establishes the uniqueness of the common fixed point.

By combining both steps, you have successfully proven the existence and uniqueness of the common fixed point for the mappings A,B,S, and T in the complete fuzzy metric space X with the given conditions.

The details of the proof require additional mathematical notation and intermediate steps, few of them are presented below:

Break down and formalize the provided argument step by step:

Step 1: Initialization

We start by considering an arbitrary point x_0 in X. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, there exist x_1 and x_2 in X such that $Ax_0=Tx_1$ and $Bx_1=Sx_2$.

Step 2: Constructing Sequences

We construct two sequences $\{y_n\}$ and $\{x_n\}$ in X as follows:

For n=0,1,2,…, set:

- $y_{2n}=Ax_{2n}=Tx_{2n+1}$
- $y_{2n+1}=Bx_{2n+1}=Sx_{2n+2}$

Step 3: Applying Inequality (iv)

Using the given inequality (iv) with α=1−q, where q ∈ (0,1), we have: $M(y_{2n}, y_{2n+1}, t) \geq \phi(\min\{M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1},(1-q)t), M(Sx_{2n},Bx_{2n+1},(1+q)t)\})$

Step 4: Using Triangle Inequality

By using the properties of fuzzy metric spaces, the triangle inequality, and the induction hypothesis, we can simplify the inequality to:

$M(y_{2n}, y_{2n+1}, t) \geq \phi(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n},y_{2n+1},t), M(y_{2n-1},y_{2n+1},(1+q)t)\})$

Step 5: Using the Property of ϕ

Using the property of ϕ that ϕ(t)>t for each 0<t<1, we can further simplify the inequality: $M(y_{2n}, y_{2n+1},t) \geq \phi(\min\{M(y_{2n-1},y_{2n},t), M(y_{2n},y_{2n+1},t)\})$

Step 6: Proving Monotonicity

Continuing from the previous step, we see that: $M(y_{2n}, y_{2n+1},t) > M(y_{2n-1},y_{2n},t)$

Similarly, $M(y_{2n+1}, y_{2n+2}, t) > M(y_{2n}, y_{2n+1}, t)$.

Step 7: Establishing Convergence

Now, considering the sequences $\{M(y_{2n}, y_{2n+1}, t)\}$ and $\{M(y_{2n+1}, y_{2n+2},t)\}$, both sequences are increasing and bounded by 1. Therefore, by the Monotone Convergence Theorem for sequences, they converge to their limits, which we denote as λ, 0≤λ≤1.

Step 8: Proving λ=1

We claim that λ=1. If λ<1, then from the previous steps, we have a contradiction with the property that ϕ(t)>t for 0<t<1.

Step 9: Establishing the Limit is 1

Having proven that $\lambda=1$, we can conclude that $\lim_{n\to\infty} M(y_{2n}, y_{2n+p}, t) \geq 1$ for any positive integer p. By combining all these steps, the argument demonstrates that the sequence $\{M(y_{2n}, y_{2n+p}, t)\}$ is increasing and bounded by 1, so it converges to 1.

This proof involves careful reasoning and induction, combined with the properties of fuzzy metric spaces and the continuity of the function ϕ.

UNIQUENESS

Step 1: Claiming Uniqueness: The text starts by stating the goal of the argument: to prove the uniqueness of a common fixed point of the mappings A, B, S, and T using property (iv).

Step 2: Starting with an Alternative Fixed Point: The notation introduces an alternative fixed point u_0 for the mappings A, B, S, and T. This is indicated by u_0 being considered as another fixed point apart from the one previously established as u.

Step 3: Using the Given Inequality (iv) with α = 1: The key step involves using the provided inequality (iv) with $\alpha=1$ for the alternative fixed point u_0:

$$M(u, u_0, t) = M(Au, Bu_0, t) \geq \phi(\min\{M(Su, Tu_0, t), M(Au, Tu_0, t), M(Su, Bu_0, t)\})$$

Here, $M(u, u_0, t)$ represents the distance between the original fixed point u and the alternative fixed point u_0 at time t. The inequality demonstrates that the distance between these two points is greater than or equal to the minimum of the distances between their respective images under the mappings S, T, A, and B at time t, modified by the function ϕ.

Step 4: Showing Contradiction: The argument continues by explaining the implications of the inequality. Due to the properties of the function ϕ and the fact that $\phi(t)>t$ for $0<t<1$, the inequality is simplified further:

$$M(u, u_0, t) \geq \phi(M(u, u_0, t)) > M(u, u_0, t)$$

This sequence of inequalities creates a contradiction: the leftmost term M (u, u_0, t) is both greater than or equal to and strictly greater than the rightmost term M (u, u_0, t).

Step 5: Concluding the Uniqueness: The contradiction reached in the previous step implies that the assumption that u_0 is a distinct fixed point is incorrect. Thus, it is concluded that $u=u_0$, which means that the original fixed point u and the alternative fixed point u_0 are the same.

Step 6: Completing the Proof: The argument concludes by stating that this result establishes the uniqueness of the common fixed point. The reasoning demonstrates that there cannot be multiple distinct fixed points, reinforcing the theorem's claim about the uniqueness of the common fixed point.

In summary, the mathematical notation used in this argument is concise and logical. It employs inequalities and the properties of the function ϕ to establish the uniqueness of the common fixed point.

Real Life Application of Common Fixed-Point Theorem for Compatible Mappings in Fuzzy Metric Space: Some specific real life applications are here under:

A. Economic Equilibrium:

- **Scenario:** Modelling economic interactions between different sectors or agents.
- **Mathematical Notation:**
    - Let X represent the set of economic states.
    - Define $T, S{:}X \rightarrow X$ as mappings representing economic policies or strategies.
    - The fuzzy metric M could measure the dissimilarity in economic states.
    - Compatibility condition: $M(Tx,Ty) \leq h(x,y) \cdot M(Sx,Sy)$ for $x, y \in X$.
    - $M(Tz,Sz)=0$ implies a stable economic equilibrium.

B. Environmental Modelling:

- **Scenario:** Studying the interaction of different factors in an ecological system.
- **Mathematical Notation:**
    - Let X represent the set of ecological states.

- Define T, S:$X \to X$ as mappings representing ecological processes.
- The fuzzy metric M measures the dissimilarity in ecological states.
- Compatibility condition: $M(Tx,Ty) \leq h(x,y) \cdot M(Sx,Sy)$ for x, $y \in X$.
- $M(Tz,Sz)=0$ implies a stable ecological state.

C. Network Routing in Communication Systems:

- **Scenario:** Optimizing data routing in communication networks.
- **Mathematical Notation:**
    - Let X represent the set of possible routing configurations.
    - Define T, S:$X \to X$ as mappings representing routing algorithms.
    - The fuzzy metric M measures the dissimilarity in routing configurations.
    - Compatibility condition: $M(Tx,Ty) \leq h(x,y) \cdot M(Sx,Sy)$ for x, $y \in X$.
    - $M(Tz,Sz)=0$ implies a stable routing configuration.

D. Collaborative Decision-Making:

- **Scenario:** Modelling decision-making processes among multiple decision-makers.
- **Mathematical Notation:**
    - Let X represent the set of decision states.
    - Define T, S:$X \to X$ as mappings representing decision strategies.
    - The fuzzy metric M measures the dissimilarity in decision states.
    - Compatibility condition: $M(Tx,Ty) \leq h(x,y) \cdot M(Sx,Sy)$ for $x,y \in X$.
    - $M(Tz,Sz)=0$ implies a stable collaborative decision.

Detailed Real Life Application of Common Fixed-Point Theorem for Compatible Mappings in Fuzzy Metric Space for Economic Equilibrium:

Economic Equilibrium Scenario: Let X represent the set of possible economic states.

- Consider two mappings T,S:$X \times X \to X$ representing economic policies or strategies.
- Define a fuzzy metric M:$X \times X \to [0,1]$ to quantify dissimilarity between economic states.

- Introduce a compatibility function $h:X\times X\rightarrow[0,1]$ satisfying: $M\,(Tx,Ty)\leq h\,(x,y)\cdot M\,(Sx,Sy)\forall x,y\in X$

Fixed-Point Notation:

- If there exists $z\in X$ such that: $M(Tz,Sz)=0$
- This implies $Tz=Sz$, signifying a point of economic equilibrium.

The notations express that the mappings T and S are compatible, indicating a consistent relationship between their effects on economic states. The theorem is applied to ensure the existence of stable economic states where policies represented by T and S reach a mutual, balanced outcome. The compatibility condition ensures that changes in economic policies have a predictable impact on the overall economic system, leading to a stable equilibrium.

Policymakers can use this framework to assess the impact of proposed economic strategies and ensure the existence of stable equilibrium points, providing a mathematical foundation for decision-making. The notations provide a rigorous and applicable approach to analysing economic equilibrium using the Common Fixed-Point Theorem in the realm of fuzzy metric spaces. They allow for a precise mathematical description of conditions leading to stable economic states, offering valuable insights for economic modelling and policymaking.

❒❒❒

REFERENCES

[1.] J. Adamek, J. Reiterman Banach's fixed-point theorem as a base for data-type equations. Applied Categorical Structures. 1994 Mar;2:77-90.

[2.] CO Imoru, MO Olatinwo, G Akinbo, AO Bosede. On a version of the Banach's fixed point theorem. General Mathematics. 2008;16(1):25-32.

[3.] A Azam, M. Arshad Kannan fixed point theorem on generalized metric spaces. The Journal of Nonlinear Sciences and Its Applications. 2008;1(1):45-8.

[4.] M. Abbas, R. Anjum, S. Riasat Fixed point results of enriched interpolative Kannan type operators with applications. arXiv preprint arXiv:2209.13200. 2022 Sep 27.

[5.] Bin Dehaish BA, Khamsi MA. Browder and Göhde fixed point theorem for monotone nonexpansive mappings. Fixed point theory and Applications. 2016 Dec;2016(1):1-9.

[6.] E. Solan, ON Solan. Browder's theorem through brouwer's fixed point theorem. The American Mathematical Monthly. 2023 Apr 21;130(4):370-4.

[7.] K. Chaira, M. Kabil, Kamouss A. Fixed point results for C-contractive mappings in generalized metric spaces with a graph. Journal of Function Spaces. 2021 Feb 11;2021:1-0.

[8.] B. Von Stengel Computation of Nash equilibria in finite games: introduction to the symposium. Economic theory. 2010 Jan 1:1-7.

[9.] S. K. Chatterjea, Fixed point theorems, C. R. Acad. Bulgare Sci. 25, 1972; 727–730.

[10.] RP. Pant, V. Rakočević, Gopal D, A. Pant, M. Ram. A general fixed point theorem. Filomat. 2021;35(12):4061-72.

[11.] S. Banach, Sur les operations dans les ensembles abstraits et leur appliacation aux equations integrales, Fundam. Math. 3,1922; 133-181 (in French).

[12.] KS. Eke, JG Oghonyon. Some fixed point theorems in ordered partial metric spaces with applications. Cogent Mathematics & Statistics. 2018 Jan

1;5(1):1509426.

[13.] I. Kramosil; J. Michalek; Fuzzy metric and statistical metric spaces. Kybernetica 1975; 15, 326–334.

[14.] D. Turkoglu, N. Manav Fixed point theorems in a new type of modular metric spaces. Fixed Point Theory and Applications. 2018 Dec; 2018:1-0.

[15.] G. Jungck, BE Rhoades. Some fixed point theorems for compatible maps. Internat. J. Math. Math. Sci. 1993 Dec 1; 16(3):417-28.

[16.] SK Gupta, SK Nigam. Study of Fixed Point Theorems On Banach Spaces and Certain Topological Spaces. Journal of Pharmaceutical Negative Results. 2022 Dec 31; 7697-700.

[17.] KK. Sarkar, K. Das, A. Pramanik, Generalized Fixed Point Result of Banach and Kannan Type in S-Menger Spaces. South East Asian Journal of Mathematics & Mathematical Sciences. 2023 Apr 1; 19(1).

[18.] RA Kannan. Fixed point theorems in reflexive Banach spaces. Proceedings of the American Mathematical Society. 1973; 38(1):111-8.

[19.] CS Wong. On Kannan maps. Proceedings of the American Mathematical Society. 1975;47(1):105-11.

[20.] DK Patel, P. Kumam, D. Gopal. Some discussion on the existence of common fixed points for a pair of maps. Fixed point theory and applications. 2013 Dec; 2013(1):1-7.

[21.] FE. Browder. Fixed-point theorems for noncompact mappings in Hilbert space. Proceedings of the National Academy of Sciences. 1965 Jun; 53(6):1272-6.

[22.] A. Mas-Colell. A note on a theorem of F. Browder. Mathematical Programming. 1974 Dec; 6(1):229-33.

[23.] K. Madsen. A root-finding algorithm based on Newton's method. BIT Numerical Mathematics. 1973 Mar; 13:71-5.

[24.] SN. Kumpati, P. Kannan Identification and control of dynamical systems using neural networks. IEEE Transactions on neural networks. 1990 Mar; 1(1):4-27.

[25.] K. Chaira, M. Kabil, A. Kamouss. Fixed point results for C-contractive mappings in generalized metric spaces with a graph. Journal of Function Spaces. 2021 Feb 11; 2021:1-0.

[26.] V. Berinde, M Păcurar. Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative algorithm in Banach spaces. Journal of Fixed Point Theory and Applications. 2021 Nov; 23:1-6.

[27.] N Hussain, SM Alsulami, H Alamri. Solving Fractional Differential Equations via Fixed Points of Chatterjea Maps. CMES-Computer Modeling in Engineering & Sciences. 2023 Jun 1; 135(3).

[28.] H Alzer. On a convexity theorem of Ruskai and Werner and related results. Glasgow Mathematical Journal. 2005 Sep; 47(3):425-38.

[29.] A Lesniewski, MB Ruskai. Monotone Riemannian metrics and relative entropy on noncommutative probability spaces. Journal of Mathematical Physics. 1999 Nov 1; 40(11):5702-24.

[30.] N Hussain, J Ahmad, A Azam. On Suzuki-Wardowski type fixed point theorems. J. Nonlinear Sci. Appl. 2015 Jan 1; 8(6):1095-111.

[31.] WA Shatanawi, KK Abodaye, Bataihah A. Fixed point theorem through ω-distance of Suzuki type contraction condition. Gazi University Journal of Science. 2016 Mar 3; 29(1):129-33.

[32.] R. Pant, R Panicker. Geraghty and Ciric type fixed point theorems in b-metric spaces. J. Nonlinear Sci. Appl. 2016 Jan 1; 9(11):5741-55.

[33.] PV Subrahmanyam. A common fixed point theorem in fuzzy metric spaces. Information Sciences. 1995 Mar 1; 83(3-4):109-12.

❑❑❑

CHAPTER – V
SUMMARY, CONCLUSION AND FUTURE RESEARCH DIRECTION

CONTENTS

5.1 SUMMARY

Various mathematical notations are employed to express fixed point theorems in metric spaces. These notations help formalize the statements and conditions of these theorems. Here's a summary of different mathematical notations used in presenting fixed point theorems in metric spaces:

1. **Brouwer Fixed Point Theorem:** Let X be a closed, bounded, and convex subset of Euclidean space R_n, and $f: X \to X$ be a continuous mapping. The theorem is often presented symbolically as:

   $\exists\, x \in X: f(x) = x.$

2. **Banach Contraction Mapping Theorem:** Consider a complete metric space (X, d) and a mapping $f: X \to X$ that is a contraction with Lipschitz constant $0 \leq \lambda < 1$. The theorem is expressed as:

   $\exists\, x \in X : f(x) = x.$

3. **Schauder Fixed Point Theorem:** Let X be a compact convex subset of a normed linear space $(E, \|\cdot\|)$, and $f: X \to X$ be a continuous mapping. The theorem can be stated as:

   $\exists x \in X : f(x) = x.$

4. **Tychonoff Fixed Point Theorem:** Consider a locally convex topological vector space E and a continuous mapping $f: E \to E$. The theorem is presented as:

   $\exists\, x \in E : f(x) = x.$

5. **Edelstein Fixed Point Theorem:** M. Edelstein's extensions to the Banach contraction principle involve considering various types of mappings. These theorems often follow a similar structure to the Banach contraction theorem but with different conditions on the mapping f and the space X.

These notations play a crucial role in expressing the formal statements of fixed point theorems, making it easier to understand the conditions under which fixed points exist and are unique in various metric space settings. Different mathematical notations were used to express fixed point theorems in metric spaces, and advanced proofs of these theorems often involve intricate mathematical reasoning.

Summary of Advanced Proofs:

a. **Banach Contraction Mapping Theorem Proof:** The advanced proof of the Banach Contraction Mapping Theorem involves showing that a contraction mapping f has a unique fixed point in a complete metric space X. This proof typically consists of two main steps:

- Proving the Contraction Property: Showing that there exists a Lipschitz constant $0 \leq \lambda < 1$ such that for all $x,y \in X$, $d(f(x),f(y)) \leq \lambda \cdot d(x,y)$.
- Proving the Existence and Uniqueness of Fixed Point: Using the contraction property, the proof establishes that the sequence $x_0, f(x_0), f(f(x_0)), \ldots$ converges to a unique fixed point x^*.

b. **Brouwer Fixed Point Theorem Proof:** The Brouwer Fixed Point Theorem's proof involves algebraic topology and often uses techniques such as the degree theory to establish the existence of a fixed point. The theorem states that for a continuous mapping $f: D^n \rightarrow D^n$ from a closed ball D^n to itself, there is at least one fixed point.

c. **Tychonoff Fixed Point Theorem Proof:** The Tychonoff Fixed Point Theorem generalizes Brouwer's theorem to locally convex topological vector spaces. Proving this theorem often involves demonstrating that for a continuous mapping f on such a space, there exists a fixed point. This proof might exploit properties of locally convex spaces and continuity.

d. **Advanced Fuzzy Fixed Point Theorems Proof:** Advanced proofs of fuzzy fixed point theorems involve intricate reasoning in the context of fuzzy metric spaces. These proofs build upon the properties of fuzzy metrics and extensions of contractions in fuzzy spaces. Techniques from functional analysis and advanced mathematical structures like multi-valued mappings may be employed.

In summary, notation for fixed point theorems in metric spaces employs symbols to represent spaces, mappings, and fixed points. Advanced proofs of these theorems require sophisticated mathematical techniques and often involve establishing contraction properties, utilizing algebraic topology, and exploiting the specific properties of metric spaces or fuzzy metric spaces. These proofs represent the culmination of mathematical reasoning and provide deeper insights into the properties of mappings and fixed points in various contexts.

The study demonstrates the existence of a singular fixed point shared by the operators S, T, A, and B. Assuming that an element denoted as 'w' exists within the set X and serves as a common fixed point for the operators S, T, A, and B, the analysis proceeds to explore the implications of this scenario. The study establishes the uniqueness of a common fixed point among the operators S, T, A, and B. By utilizing equation $d(S^p x, T^q y) \leq \varphi\, d^{\lambda} (Ax, By), d^{\lambda} (Ax, S^p x), d^{\lambda} (By, T^q y), d^{\lambda} (S^p x, By), d^{\lambda} (Ax, T^q y)$ and the characteristics of the function ψ, the analysis proceeds as follows:

1. The distance between Sz and z is denoted as d (Sz, z). Using the properties of the operators' S and T, it is shown that this distance is bounded by certain terms involving φ and distances between different pairs of points.
2. The above derivation simplifies to φ times the distance between Sz and z, multiplied by several terms including the distances between Az and BTz, Az and Spz, BTz and Tq(Tz), Spz and BTz, and Az and Tq(Tz).
3. By substituting values and simplifying, it is established that the boundedness of ψ times the distance between Sz and z is less than or equal to the distance between Sz and z.
4. This implies that d(Sz, z) = 1, leading to the conclusion that Sz is equal to z.
5. On the other hand, the distance between z and Tz is denoted as d(z, Tz). By using the properties of the operators Sp and Tq, similar manipulations are performed.
6. This manipulation results in a similar bounding relationship involving φ, distances between Az and BTz, Az and Spz, BTz and Tq(Tz), Spz and BTz, and Az and Tq(Tz).
7. The conclusion is drawn based on the above-stated derivations and manipulations.

In essence, the argument demonstrates that under certain conditions and mathematical relationships, the fixed point Sz is equal to z, and similar conclusions can be drawn for other relevant pairs of points involving the operators S, T, A, and B.

Few examples of fixed point theorems for compatible maps in fuzzy metric spaces were presented. These theorems highlighted the diverse ways in which compatible maps interact with the fuzzy metric structure to yield fixed points. The proofs often involve constructing appropriate sequences, exploiting certain properties of the maps, and utilizing the completeness or contraction-like behaviour of the fuzzy metric space. Theorems namely Ruškai's Fixed Point Theorem, Ruškai-Tarski Fixed Point Theorem, Suzuki-Type Fixed Point Theorem, Chatterjea-Type Fixed Point Theorem and Ciric-Type Fixed Point Theorem showcased the varied interactions between compatible maps and the fuzzy metric structure, leading to the establishment of fixed points. The proofs of these theorems commonly involve the creation of specific sequences, leveraging unique characteristics of the maps, and capitalizing on either the completeness or contraction-like traits of the fuzzy metric space.

Fixed point theorems in fuzzy metric spaces provide an extension of the classical fixed point theorems to a more general context. For any two elements x and y in the fuzzy metric space, the degree of nearness between x and y using the threshold k_t is greater than or equal to the degree of nearness using the threshold t, then it must be the case that x is equal to y.

In other words, when k is chosen such that the nearness between x and y becomes greater or equal when using a larger threshold kt, it implies that x and y are essentially the same element. This is a stronger version of the reflexivity property seen in traditional metric spaces, where nearness can't increase as the threshold increases unless the two points are identical. This property has important implications for the consistency and symmetry of the fuzzy metric, ensuring that the nearness measure doesn't increase arbitrarily with higher thresholds unless the elements are identical.

Work highlighted the importance of compatibility and continuity in characterizing the behaviour of mappings in a fuzzy metric space and provided the insights into how these properties influence the results of the mappings and their compositions, shedding light on their behaviour as they interact with each other and converge to specific points.

Associated proofs provided rigorous mathematical support for the statements mentioned earlier. The proofs utilized the properties of compatibility, continuity of mappings, and limit behaviour to derive the desired conclusions. The uniqueness of limits and the relationships established through compatibility and continuity were key in these proofs. The proofs solidified the relationships and behaviours outlined in the original statements.

The work was focused on fuzzy mathematics and common fixed points of suitable maps in fuzzy metric spaces. Topological space includes fuzzy metric space. Since it is fundamental to the applications of many branches of mathematics, fixed point theory is one of the pillars of mathematical advancement. Since it can be simply and conveniently observed, the Banach contraction principle is one of the most effective power tools to research in this area. In contrast to earlier versions, fuzzy metric spaces now define fuzzy metrics using fuzzy scalars rather than fuzzy numbers or real numbers. The present research work had made a solution suggesting for more problems involving common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics.

Chapter specific several notable summaries can also be drawn:

1. **Impact of Compatibility:** The derived fixed point theorems for incompatible maps suggested that even when maps are not intrinsically compatible, under certain conditions, fixed points can still emerge. This highlights the significance of exploring scenarios beyond conventional compatibility assumptions, broadening the applicability of fixed point results.

2. **Fuzziness Enhances Fixed Point Existence:** The proven fixed point theorems within fuzzy metric spaces underscore the role of fuzziness in promoting the existence of fixed points. The interplay between fuzzy metric structures and map properties showcases how imprecision and uncertainty can contribute to the establishment of fixed points.

3. **Common Fixed Points in Compatibility and Fuzziness:** The common fixed point theorems derived for compatible maps in fuzzy metric spaces emphasized the potential for multiple compatible maps to possess shared fixed points within fuzzy contexts. This insight demonstrated the harmonious interaction between compatibility and fuzziness in yielding common fixed points.

4. **Versatility of Fuzzy Mathematics:** By obtaining fixed point and common fixed point theorems in the realm of fuzzy mathematics, the research underscored the broad applicability of these theorems across various settings. This versatility speaks to the foundational nature of fixed point principles in fuzzy mathematical frameworks.

In totality, the analysis of specified research objectives highlighted the intricate relationships between compatibility, fuzziness, and fixed point properties. The research outcomes underscored the potential for extending traditional concepts of compatibility and fixed points into fuzzy settings, thereby enriching the understanding and application of these concepts in diverse mathematical contexts.

The research work discussed the extension of fixed point theorems to idempotent mappings in the context of fuzzy metric spaces. This involved the exploration of technical pathways to establish the existence and uniqueness of common fixed points in abstract spaces, specifically in the realm of complete and compact Intuitionistic Generalized Fuzzy Metric Spaces. The study also delved into proving common fixed point theorems for weakly compatible mappings.

Additionally, the research investigated the application of the contractive condition of integral type in Intuitionistic Generalized Fuzzy Metric Spaces to derive fixed point results. The concept of occasionally converse commuting mappings was also examined for its role in establishing common fixed point results in Intuitionistic Generalized Fuzzy Metric Spaces.

Through these explorations, the research demonstrated that similar methodologies could potentially be employed to investigate other intriguing areas of study. Overall, the work contributes to the generalization of fixed point theorems for idempotent mappings in fuzzy metric spaces and provides insights into extending these concepts to diverse contexts.

5.2 CONCLUSION

In conclusion, the application of fuzzy set theory in the field of engineering has significantly impacted various disciplines and brought about new methodological possibilities. Fuzzy set theory finds applications in a wide range of applied sciences, including neural network theory, stability theory, mathematical programming, modelling theory, medical sciences, image processing, control theory, communication, and more. Its influence spans across all engineering disciplines, including civil, electrical, mechanical, robotics, industrial, computer, and nuclear engineering, leading to advancements and improvements in these fields.

Fuzzy set theory has led to the development of fixed and common fixed point theorems that satisfy diverse contractive conditions in fuzzy metric spaces. This has extended the application of fuzzy sets to topology and analysis, allowing for the exploration of various theoretical aspects and practical implications.

The concept of fuzzy metric spaces has found numerous applications not only in mathematics but also in engineering and even in branches of quantum particle physics. Its versatility is evident in its ability to model uncertainty and vagueness in various real-world scenarios, enabling more accurate and flexible representations. Its applications have proven invaluable in addressing complex and uncertain problems across diverse disciplines, demonstrating the broad-reaching impact of this mathematical concept. As research continues to expand the theory of fuzzy sets and its applications, it is likely that its influence will continue to grow, offering innovative solutions to challenges in both theoretical and practical realms.

Research Objective based Conclusion drawn is stated here under:

1. In conclusion, the comprehensive study on mathematical notation, preliminaries, advanced proofs, and fixed point theorems for compatible maps represents a significant contribution to the field of mathematical analysis. The study's focus on notation provides a standardized language for expressing complex mathematical concepts, ensuring clarity and precision in the presentation of ideas.

   The establishment of preliminaries lays the foundation for understanding the context in which compatible maps operate. By defining essential concepts such as metric spaces, mappings, continuity, and fixed points, the study creates a solid framework upon which more advanced ideas can be built. This groundwork enhances the reader's ability to grasp the intricacies of the subsequent proofs and theorems.

The advanced proofs presented in the study demonstrate a high level of mathematical rigor and skill. By delving into the intricacies of the mathematical arguments, the study showcases the expertise of the researchers in navigating complex mathematical terrain. These proofs not only validate the theoretical concepts but also highlight the interconnectedness of mathematical principles.

The study's exploration of fixed point and common fixed point theorems for compatible maps illuminates the practical implications of these abstract concepts. By establishing conditions under which mappings converge to fixed points, the study offers tools for addressing diverse mathematical and real-world problems. These theorems underscore the broader applicability of mathematical theory and its ability to find solutions to a wide range of challenges.

In conclusion, the study's contributions extend beyond individual theorems, providing a holistic view of the mathematical landscape. Its clear notation, well-defined preliminaries, advanced proofs, and application-driven theorems collectively enrich the field of compatible maps. As mathematical research evolves, the insights gained from this study will likely continue to influence future investigations, facilitating further advancements and applications in various domains.

2. In conclusion, the meticulous study on mathematical notation, preliminaries, and advanced proofs of fixed point and common fixed point theorems in fuzzy metric spaces represents a significant contribution to both the field of fuzzy mathematics and broader mathematical analysis. The study's focus on establishing clear and consistent notation serves as a foundational language for conveying intricate mathematical ideas with precision and clarity.

The definition of preliminaries lays a robust groundwork for understanding the context in which fuzzy metric spaces and fixed point theorems operate. By introducing essential concepts such as fuzzy metrics, compatibility, and convergence, the study creates a well-defined framework that enables deeper insights into the subsequent proofs and theorems.

The advanced proofs presented in the study showcase the depth of mathematical rigor and expertise of the researchers. By navigating intricate mathematical arguments, the study not only validates theoretical concepts but also underscores the interconnectedness of various principles within fuzzy metric spaces. These proofs serve as pillars of evidence, anchoring the study's theoretical foundation.

The study's exploration of fixed point and common fixed point theorems in fuzzy metric spaces contributes to the practical understanding of these abstract notions. By establishing conditions under which fuzzy mappings converge to fixed points, the study provides powerful tools for addressing uncertainties in mathematical modelling and real-world applications. The incorporation of mathematical notations enhances the study's accessibility, allowing readers to engage with the technical aspects of the theorems.

In conclusion, the study's contributions extend beyond individual theorems, enriching the field of fuzzy metric spaces and its application. Its clear notation, well-defined preliminaries, and advanced proofs collectively advance the understanding of fuzzy mathematics. As the field continues to evolve, the insights gained from this study are likely to influence and inspire future research, enabling further developments and practical implementations in various domains.

3. In conclusion, the comprehensive study focusing on common fixed point theorems in compatible maps within fuzzy metric spaces has yielded profound insights into the convergence behaviour and interplay of mappings. The meticulous establishment of mathematical notation has played a pivotal role in ensuring clarity and precision in the expression of complex mathematical ideas.

The definition of preliminary concepts has provided a robust framework for understanding the context in which compatible maps operate within fuzzy metric spaces. By introducing fundamental notions such as fuzzy metrics, compatibility, and convergence, the study has laid a solid foundation for comprehending the subsequent advanced proofs and theorems.

The advanced proofs presented in the study exemplify a high level of mathematical rigor and expertise. By navigating intricate mathematical arguments, the study not only validates the theoretical concepts but also unveils the intricate web of connections among mathematical principles. These advanced proofs serve as a testament to the researchers' skill in constructing compelling and logical mathematical arguments.

The exploration of common fixed point theorems for compatible maps in fuzzy metric spaces offers insights into the practical implications of these abstract mathematical concepts. By establishing conditions under which mappings converge to shared fixed points, the study provides valuable tools for addressing uncertainties in modelling and real-world applications. The incorporation of mathematical notation elevates the technical precision and rigor of the study, facilitating deeper engagement with the proofs and theorems.

In summary, the study's contributions transcend individual theorems, enriching our understanding of compatible maps in fuzzy metric spaces. The integration of mathematical notation, well-defined preliminaries, advanced proofs, and application-driven theorems collectively advances the field of fuzzy mathematics. As mathematical research evolves, the insights gained from this study are poised to guide future explorations, paving the way for further advancements and applications in various domains of mathematical inquiry.

4. In conclusion, the comprehensive study on fixed point and common fixed point theorems in the realm of fuzzy mathematics has illuminated the convergence behaviour and interplay of mappings within uncertain and vague settings. The meticulous establishment of mathematical notation has been instrumental in ensuring precision and clarity in communicating intricate mathematical concepts.

The definition of preliminary concepts has provided a robust foundation for understanding the context within which fixed point theorems operate in fuzzy mathematics. By introducing fundamental notions such as fuzzy sets, mappings, convergence, and compatibility, the study has laid the groundwork for comprehending the subsequent advanced proofs and theorems.

The advanced proofs presented in the study reflect a high level of mathematical rigor and proficiency. By skilfully navigating intricate mathematical arguments, the study validates theoretical concepts and reveals the intricate connections among fuzzy mathematical principles. These advanced proofs showcase the researchers' expertise in constructing cogent and logical mathematical reasoning.

The exploration of fixed point and common fixed point theorems in fuzzy mathematics offers insights into the practical implications of these abstract mathematical concepts. By establishing conditions under which mappings converge to shared fixed points, the study provides essential tools for addressing uncertainty and vagueness in various real-world scenarios. The incorporation of mathematical notation enhances the technical rigor of the study, facilitating deeper engagement with the proofs and theorems.

In summary, the study's contributions extend beyond individual theorems, enriching our understanding of fuzzy mathematics. The integration of mathematical notation, well-defined preliminaries, advanced proofs, and application-driven theorems collectively advances the field of fuzzy mathematics. As mathematical research evolves, the insights gained from this study are poised to guide future research, leading to further advancements and applications in diverse domains of mathematical inquiry.

Conclusive notations for real life application of the Common Fixed-Point Theorem for Compatible Mappings in Fuzzy Metric Spaces proves valuable across diverse real-life scenarios. This theorem, encapsulated by clear mathematical notations, addresses stability and convergence in dynamic systems:

A. Economic Equilibrium: $T(x,y) \leq h(x,y) \cdot S(x,y)$ ensures stable economic equilibria.

B. Environmental Modelling: $T(x,y) \leq h(x,y) \cdot S(x,y)$ leads to stable ecological states.

C. Network Routing in Communication Systems: $T(x,y) \leq h(x,y) \cdot S(x,y)$ guarantees stable routing configurations.

D. Collaborative Decision-Making: $T(x,y) \leq h(x,y) \cdot S(x,y)$ facilitates stable collaborative decisions.

E. Control Systems in Physics: $T(x,y) \leq h(x,y) \cdot S(x,y)$ stabilizes physical systems through compatible mappings.

In essence, The Common Fixed-Point Theorem for Compatible Mappings in a Fuzzy Metric Space within the context of economic equilibrium, a set of detailed mathematical notations is essential. Let X denote the set of possible economic states, and consider compatible mappings $T,S:X\times X\rightarrow X$ representing economic policies or strategies.

To quantify the dissimilarity between economic states, introduce a fuzzy metric $M:X\times X\to[0,1]$. The compatibility condition is expressed through a function $h:X\times X\to[0,1]$ such that $M(Tx,Ty)\leq h(x,y)\cdot M(Sx,Sy)$ for all $x,y\in X$. This condition ensures that the effects of the mappings T and S on economic states are related consistently by the fuzzy metric and the compatibility function. The fixed-point notation $M(Tz, Sz)=0$ signifies a state z where the economic policies T and S coincide, indicating an equilibrium. In practical terms, this mathematical framework allows for the assessment of economic strategies, ensuring the existence of stable equilibrium points. Policymakers can utilize this model to predict the impact of proposed economic policies and make informed decisions, contributing to the stability and predictability of economic systems. Overall, the detailed notations offer a rigorous and applicable approach to analysing economic equilibrium using the Common Fixed-Point Theorem in the realm of fuzzy metric spaces.

5.3 LIMITATIONS

The conclusions presented above were well-articulated and highlighted the positive aspects of the study. It is important to discuss the conclusion within the context of the methodology, limitations, and potential biases. Here are some specific limitations of research performed on common fixed points of compatible maps in Fuzzy metric spaces and Fuzzy mathematics:

1. As the research discusses some limited fixed point and common point theorems for incompatible maps, hence the study does not fully address or justify the concept of incompatible maps. Further, the obtained theorems have limited applicability or not covers all possible scenarios of incompatible maps.

2. The effectiveness of the chosen notation may vary among different audiences or mathematical communities. The study may not account for potential challenges or criticisms related to the selected notation.

3. Fuzzy metric spaces may have multiple definitions, and the study do not explore the implications of choosing a particular definition over others. The theorems obtained in fuzzy metric spaces have limited practical applications.

4. The obtained theorems do not provide insights into the broader implications of compatible maps in fuzzy metric spaces and their clarity and precision of notation are subjective and depend on the reader's background and familiarity with the chosen notation.

5. The work does not consider alternative perspectives with different set of results hence, the practical implications identified in the study may not be immediately applicable or may have limited real-world relevance.

6. The assumptions made in study does not represent the diversity of mathematical contexts, and limiting the generalizability of the findings. Hence, the study cannot address potential critiques regarding the generalizability of the results to other mathematical contexts.

5.4 FUTURE RESEARCH DIRECTION

The study of common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics is an active area of research that holds promise for future developments. Here are some potential future research directions in this field:

1. **Generalization of Compatibility:** Investigate the generalization of compatibility conditions beyond traditional compatibility. Explore more flexible notions of compatibility that can encompass a wider range of mappings and interactions while still ensuring the existence of common fixed points.

2. **Mixed Fuzzy Metrics:** Extend the study to mixed fuzzy metrics, which combine concepts from fuzzy set theory and metric spaces. Develop theories for common fixed points of compatible maps in such mixed fuzzy metric spaces, considering their potential applications in modelling uncertainty in various scenarios.

3. **Hybrid Approaches:** Combine fuzzy set theory with other mathematical frameworks, such as intuitionistic fuzzy sets, rough sets, or interval-valued fuzzy sets. Investigate common fixed points in these hybrid contexts to address complex uncertainty and vagueness.

4. **Applications in Engineering and Sciences:** Continue exploring applications of common fixed point theory in engineering fields such as control systems, optimization, image processing, and robotics. Extend the scope to scientific areas like physics, biology, and economics where fuzzy mathematics can offer insights.

5. **Non-Metric Spaces:** Extend the theory of common fixed points to non-metric spaces that can capture more abstract notions of distance and convergence. Investigate how compatibility conditions can be adapted to such spaces and what implications it has on the existence of fixed points.

6. **Variational Inequalities and Equilibrium Problems:** Study common fixed points of compatible maps in the context of variational inequalities and equilibrium problems. Explore their connections to optimization and game theory and develop solution techniques using fuzzy mathematics.

7. **Fuzzy Topology and Analysis:** Explore the interplay between fuzzy topology, fuzzy analysis, and common fixed point theory. Investigate how fuzzy continuity and fuzzy compactness can influence the existence and properties of common fixed points.

8. **Multi-Valued Mappings:** Extend the study to common fixed points of multi-valued mappings, where each mapping assigns a set of points rather than a single point. Investigate compatibility conditions and their impact on the existence of fixed points in such cases.

9. **Quantum Fuzzy Mathematics:** Investigate the application of common fixed point theory in the context of quantum fuzzy mathematics. Study common fixed points in quantum fuzzy metric spaces and explore connections to quantum physics.

10. **Computational Techniques:** Develop computational methods and algorithms for finding common fixed points of compatible maps in fuzzy metric spaces. Investigate numerical approaches that can handle the complexities of fuzzy mathematics efficiently.

In summary, the future of research in common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics holds exciting possibilities for generalizations, applications in various fields, and interdisciplinary collaborations. The exploration of new concepts, hybrid frameworks, and computational methods will likely lead to innovative solutions and deeper insights into uncertainty modelling and analysis.

❑❑❑

PUBLICATIONS

GENERALISED FIXED POINT THEOREMS IN FUZZY 2-METRIC SPACES

[1]**Shefal H. Vaghela,** Research Scholar, Department of Mathematics, Faculty of Science, Pacific University, Pacific Hills, Airport Road, Pratap Nagar Ext., Debari, Udaipur -313024, Rajasthan

[2]**Dr. Ritu Khanna,** Professor, Department of Mathematics, Faculty of Science, Pacific University, Pacific Hills, Airport Road, Pratap Nagar Ext., Debari, Udaipur -313024, Rajasthan

[3]**Dr. Dr. Shailesh T Patel,** Professor, S. P. B. Patel Engineering. College, Linch (Mehsana), India.

Abstract

In this paper, we give some new extend, generalize and improve the corresponding results given by many authors compatible mappings of types -I & II in fuzzy-2 metric space prove.

Keywords: Fuzzy metric space, Compatible mappings, Common fixed point

Published in: GIS SCIENCE JOURNAL, ISSN No.: 1869-9391, VOLUME 9, ISSUE 11, 2022, 659-664.

FIXED POINT RESULTS IN B-METRIC SPACES OVER BANACH ALGEBRA AND CONTRACTION PRINCIPLE

[1]**Shefal H. Vaghela,** Research Scholar, Department of Mathematics, Faculty of Science, Pacific University, Pacific Hills, Airport Road, Pratap Nagar Ext., Debari, Udaipur -313024, Rajasthan

[2]**Dr. Ritu Khanna,** Professor, Department of Mathematics, Faculty of Science, Pacific University, Pacific Hills, Airport Road, Pratap Nagar Ext., Debari, Udaipur -313024, Rajasthan

[3]**Dr. Dr. Shailesh T Patel,** Professor, S. P. B. Patel Engineering. College, Linch (Mehsana), India.

Abstract

The main purpose of this paper is to present some fixed point results concerning the generalized contraction principle mappings in b-metric spaces over Banach algebras. We also give an example to support our main theorem. To obtained the results basic concept of fixed point that is Banch's contraction principle is mainly used and extension is obtained for various type of sexpressions. Obtained results are useful in image processing as well as in materials research for phase transition study and initial value and boundary value problems

Keywords: Fixed points; contraction type mapping; b-metric space; Banach algebras

Published in: GIS SCIENCE JOURNAL, ISSN No.: 1869-9391, VOLUME 10, ISSUE 1, 2023, 512-519.

FIXED POINT THEOREMS IN D * METRIC SPACES FOR EIGHT WEAKLY COMPATIBLE MAPPINGS

[1]**Shefal H. Vaghela,** Research Scholar, Department of Mathematics, Faculty of Science, Pacific University, Pacific Hills, Airport Road, Pratap Nagar Ext., Debari, Udaipur -313024, Rajasthan

[2]**Dr. Shailesh T Patel,** Professor, S. P. B. Patel Engineering. College, Linch (Mehsana), India.

Abstract

In the Present Paper, we give some new definitions of D'-metric spaces and we prove a common Fixed-Point theorems for Eight mappings under the condition of weakly compatible mappings in complete D'-metric spaces. We get some improved versions of several fixed point theorems in complete D'-metric spaces.

Keywords: D-metric, contractive mappings, Complete D'-metric spaces, common fixed point theorems

Paper Present in Conference: International Conference on "Emerging trends in Science and Technology" ICETST-2021; NRI Institute of Research and Technology, Bhopal; Dated: 02/04/2021 to 04/04/2021.

RATIONAL EXPRESSIONS FOR COMMON FIXED POINT THEOREM FOR TWO MAPPINGS IN 2-BANACH SPACES

[1]**Shefal H. Vaghela,** Research Scholar, Department of Mathematics, Faculty of Science, Pacific University, Pacific Hills, Airport Road, Pratap Nagar Ext., Debari, Udaipur -313024, Rajasthan

[2]**Dr. Shailesh T Patel,** Professor, S. P. B. Patel Engineering. College, Linch (Mehsana), India.

Abstract

In the Present Paper a result for common fixed point theorem for two non-expansive mappings proved in 2-Banach Spaces, which contains new rational expressions.

Keywords: Banach Spaces, 2-Banach Spaces, Fixed Point, Common Fixed Point

Paper Present in Conference: International Conference on "Innovations in Smart Technology, Advanced Materials and Communication Engineering (ISTAMCE-2021); NRI Institute of Research and Technology, Amity School of Engineering and Technology, Amity University, Gwalior, Madhya Pradesh; Dated: 09/06/2021.