CHAPTER – IV

FIXED POINT THEOREM IN COMPATIBLE MAPPING

CONTENTS

4.1 INTRODUCTION

Fixed point theorems are important results in mathematics that deal with the existence of points that remain unchanged under certain mappings or transformations. These theorems have numerous applications in various fields, including mathematics, economics, computer science, and physics. A fixed point of a function f is a point x such that f (x) = x.

When it comes to compatible maps, fixed point theorems can be applied in scenarios where multiple mappings are involved and there's a need to establish the existence of common fixed points or compatible fixed points under certain conditions. Compatible maps are usually maps that satisfy certain compatibility conditions with each other. Here are a few fixed point theorems that involve compatible maps:

1. **Banach's Fixed Point Theorem:** This theorem is one of the most well-known fixed point theorems. It states that if X is a complete metric space and T: X→X is a contraction mapping (i.e., there exists a constant $0 \leq k < 1$ such that $d(Tx, Ty) \leq k \cdot d(x, y)$ for all x, y ∈ X, then T has a unique fixed point[1, 2].

2. **Kannan's Fixed Point Theorem:** Kannan's theorem extends Banach's theorem to the case of a self-map T on a complete metric space X satisfying a certain weak contractive condition. The condition requires that for all x ∈ X, there exists a sequence (x_n) such that T_{x_n} converges to T_x and $d\left(T_{x_n}, T_{x_{n+1}}\right) \leq d\left(x_n, x_{n+1}\right)$ [3, 4].

3. **Browder's Fixed Point Theorem:** This theorem deals with a set of compatible maps on a nonempty, convex, and closed subset of a Banach space. It states that if the maps satisfy certain conditions, then they have a common fixed point[5, 6].

4. **Rosenberg-Kannan Fixed Point Theorem:** This theorem considers a finite family of self-maps on a metric space and provides conditions under which there exists a unique point that is a fixed point for each map in the family[7, 8].

5. **Chatterjea's Fixed Point Theorem:** Chatterjea's theorem generalizes the concept of compatible maps. It establishes the existence of a common fixed point for a finite family of maps that satisfy a weak commutativity condition[9, 10].

These theorems, among others, demonstrate the power of fixed point arguments in establishing the existence of solutions in various mathematical and practical contexts. The concept of compatible maps adds an additional layer of structure to the mappings being considered, allowing for more intricate results to be derived.

Fixed point theorems involving compatible maps are powerful tools in various mathematical spaces. Here are some instances of such theorems in different spaces:

1. Banach Spaces: Banach spaces are complete normed vector spaces, where the norm satisfies the triangle inequality. A common fixed point theorem for compatible maps in Banach spaces can be a generalized version of Banach's Fixed Point Theorem, where the maps are compatible and satisfy a contraction condition[11]:

Let $(X, \|\cdot\|)$ be a complete Banach space, and let $f: X \to X$ be a compatible map. If there exists a constant $0 < k < 1$ such that for all x, y in X:

$$\|f(x) - f(y)\| \leq k * \|x - y\|$$

then f has a unique fixed point x^* in X.

2. Partial Metric Spaces: A partial metric space[12] is similar to a metric space, but the distance between distinct points can be zero. Common fixed point theorems for compatible maps in partial metric spaces adapt the contraction condition accordingly:

Let (X, p) be a complete partial metric space, and let $f: X \to X$ be a compatible map. If there exists a constant $0 < k < 1$ such that for all x, y in X:

$$p(f(x), f(y)) \leq k * p(x, y)$$

then f has a unique fixed point x^* in X.

3. Probabilistic Metric Spaces: Probabilistic metric spaces generalize metric spaces by allowing distances to be probabilistic[13]. Fixed point theorems involving compatible maps in probabilistic metric spaces consider compatibility in terms of probabilities:

Let (X, d, P) be a probabilistic metric space, where d is the probabilistic distance and P is the underlying probability distribution. If f: X → X is a compatible map in terms of probabilities, then there exist fixed points for f under suitable conditions.

4. Quasi Metric Spaces: Quasi metric spaces relax the triangle inequality, allowing for a weaker form of the metric axioms[14]. Fixed point theorems for compatible maps in quasi metric spaces take into account the modified compatibility condition:

Let (X, q) be a quasi-metric space, and let f: X → X be a compatible map. If there exists a constant $0 < k < 1$ such that for all x, y in X:

$q(f(x), f(y)) \leq k * q(x, y)$

then f has a unique fixed point x* in X.

These are just a few examples of how compatible maps and fixed point theorems can be adapted to various mathematical spaces. The key idea remains the same: under suitable conditions, compatible maps will have fixed points that satisfy certain properties related to the metric or space being considered. The choice of space depends on the problem at hand and the specific mathematical structures involved.

4.1.1 Banach's Fixed Point Theorem

Banach's Fixed Point Theorem[1, 2], also known as the Contraction Mapping Theorem, states the following:

Given a complete metric space X with a metric d, and a self-mapping T: X→X, if T is a contraction mapping with a contraction constant $0 \leq k < 1$, then there exists a unique fixed point $x*$ in X such that $T(x*) = x*$.

Contractive Mapping: A mapping T: X→X on a metric space X is contractive if there exists a constant $0 \leq k < 1$ such that for all x, y ∈ X, the distance between their images under T is at most k times the distance between x and y: $d(T(x), T(y)) \leq k \cdot d(x,y)$

Fixed Point: Given a mapping T:X→X, a point $x*$ in X is a fixed point of T if $T(x*) = x*$.

Compatible Maps: Consider two self-mappings of a metric space X: T: X→X and S: X→X. The maps T and S are considered compatible if they satisfy the following condition for all $x \in X$: $T(S(x)) = S(T(x))$

Lemma 1: Compatibility Implies Common Fixed Point: If T and S are compatible self-mappings on a complete metric space X, and both T and S are contraction mappings with the same contraction constant $0 \le k < 1$, then there exists a common fixed point $x*$ for both T and S, meaning $T(x*)=x*$ and $S(x*) = x*$.

Proof Sketch for Lemma:

1. By Banach's Fixed Point Theorem, both T and S have unique fixed points xT and xS respectively.
2. Since T and S are compatible, T (xS) = S (T(xS)) = S (xT).
3. The uniqueness of fixed points implies that xT = xS, and this common point is a fixed point for both T and S.

Lemma 2 : Compatibility Preserves Fixed Point:

If T and S are compatible self-mappings on a complete metric space X, and T has a unique fixed point xT, then S (xT) is also a fixed point of T.

Proof Sketch for Lemma:

1. Using compatibility: T (S(xT)) = S (T(xT)) = S(xT).
2. Thus, S (xT) is a fixed point of T.

These lemmas highlight the relationship between compatible maps, contraction mappings, and fixed points, providing insights into how these concepts interplay within the framework of Banach's Fixed Point Theorem.

4.1.2 Banach's Fixed Point Theorem for Compatible Maps

Banach's Fixed Point Theorem is a significant result in mathematics that guarantees the existence and uniqueness of fixed points for certain types of mappings. When combined with the concept of compatible maps, it leads to interesting applications and insights. Here's an explanation of Banach's Fixed Point Theorem applied to compatible maps[15, 16]:

Suppose we have a complete metric space X with a metric d, and let T: X→X be a self-mapping of X, which means T maps from X to itself.

Furthermore, let's assume that T is a contractive mapping with a contraction constant $0 \le k < 1$. This means that for any two points x, y ∈ X, the distance between their images under T is at most k times the distance between x and y:

$d\ (T(x),\ T(y)) \le k \cdot d\ (x,y)$

Now, let S: X→X be another self-mapping of X. The maps T and S are considered compatible if the following condition holds for all points x ∈ X:

$T\ (S(x)) = S\ (T(x))$

Before delving into Banach's Fixed Point Theorem for compatible maps, let's cover some preliminary concepts that are essential for understanding the theorem:

1. **Metric Space:** A metric space is a mathematical structure consisting of a set X and a distance function (metric) d: $X \times X \to \mathbb{R}$ that satisfies certain properties. The distance function d (x, y) measures the distance between two points x and y in the metric space. It is required to be non-negative, symmetric (d (x, y) = d(y, x)), and satisfy the triangle inequality ($d(x, z) \le d(x, y) + d(y, z)$).
2. **Self-Map (Function):** A self-map or function is a mapping from a set to itself. In the context of metric spaces, a self-map f: X → X is a function that takes elements from the metric space X and maps them back to X.
3. **Fixed Point:** A fixed point of a function f: X → X is a point x ∈ X such that f(x) = x. In other words, it's a point that is unchanged under the action of the function.
4. **Contraction Mapping:** A self-map f: X → X is a contraction mapping if there exists a constant $0 < k < 1$ such that for all x and y in X:

   $D\ (f(x),\ f(y)) \le k * d\ (x,\ y)$

   In other words, a contraction mapping reduces distances between points. This concept is crucial for understanding Banach's Fixed Point Theorem.
5. **Complete Metric Space:** A metric space (X, d) is complete if every Cauchy sequence in X converges to a limit that is also in X. In other words, there are no

"missing points" in the space; all limits of Cauchy sequences are within the space itself.

6. **Banach's Fixed Point Theorem:** Banach's Fixed Point Theorem, also known as the Contraction Mapping Principle, states that if (X, d) is a complete metric space and f: X → X is a contraction mapping, then f has a unique fixed point in X. This fixed point can be found by repeatedly applying the function to any starting point and observing the convergence of the resulting sequence.

7. **Compatible Maps (for the specialized version):** Two self-maps f and g defined on a metric space (X, d) are compatible if, for all x and y in X, the following inequality holds:

   $d(f(x), f(y)) \leq d(g(x), g(y))$

   This compatibility condition ensures that the distances between the images of any two points under the map f do not increase more than the distances between their images under the map g.

With these preliminary concepts in mind, we can now move on to discussing Banach's Fixed Point Theorem for compatible maps.

Theorem: Banach's Fixed Point Theorem for Compatible Maps

If X is a complete metric space and T: X→X is a contractive mapping with a contraction constant $0 \leq k < 1$, and S: X→X is a compatible mapping with T, then both T and S have unique fixed points in X.

Proof Sketch:

1. **Existence of Fixed Points:** By Banach's Fixed Point Theorem, since T is contractive, it has a unique fixed point xT in X. Similarly, since S is compatible with T, it has a unique fixed point xS in X.
2. **Uniqueness of Fixed Points:** Suppose yT is another fixed point of T and yS is another fixed point of S. Using the compatibility condition, we have: T(yS) =S (T(yT)) = S (yT) However, the uniqueness of fixed points for T and S implies that yT = xT and yS = xS.
3. **Conclusion:** Thus, both T and S have unique fixed points, which are xT and xS respectively.

This theorem is useful in situations where multiple mappings interact with each other and are compatible in a certain way. It ensures the existence and uniqueness of fixed points for each mapping, which can have various applications in mathematics and its applications in other fields.

4.1.3 Kannan's Fixed Point Theorem

Kannan's Fixed Point Theorem is a result in mathematics that deals with the existence of fixed points for certain types of mappings in metric spaces[3, 17]. The theorem is named after its creator, the Indian Mathematician K. Kannan.

A fixed point of a function f is a point x in its domain such that f (x) = x. In other words, a fixed point is a point that remains unchanged under the action of the function. Kannan's Fixed Point Theorem states the following:

Theorem: Let X be a non-empty complete metric space, and let T: X→X be a self-mapping (a mapping from X to itself). If there exists a constant $0 \leq q < 1$ such that for all x, y ∈ X, the inequality $d(Tx, Ty) \leq q \cdot \max\{d(x,y), d(Tx,x), d(Ty, y)\}$ holds, where d is the metric on X, then T has a unique fixed point.

In simpler terms, this theorem provides conditions under which a self-mapping T on a complete metric space X is guaranteed to have a unique fixed point. The key requirement is that the distances between images of points under T should contract towards each other by a factor $q<1$ as the points themselves get closer. This ensures that as the process of iterating T continues, the images of points will converge to a common point, which is the unique fixed point of T.

Kannan's Fixed Point Theorem has applications in various areas of mathematics and its proofs involve concepts from metric space theory and contraction mappings. It's a fundamental result in the theory of fixed point theorems and plays an important role in analysing the convergence of iterative algorithms in various fields[17].

Kannan's Fixed Point Theorem involves several definitions and lemmas that are crucial to understanding and proving the theorem. Let's go through some of the key definitions and lemmas[18]:

1. Metric Space: A metric space is a set X equipped with a distance function $d: X \times X \rightarrow R$ that satisfies certain properties:

- $d(x,y) \geq 0$ for all $x,y \in X$, and $d(x,y) = 0$ if and only if $x = y$.
- $d(x,y) = d(y,x)$ (symmetry).
- $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

2. Self-Mapping: A self-mapping (or self-map) of a metric space X is a function $T: X \rightarrow X$ that maps elements from X to itself.

3. Contractive Mapping: A mapping T is said to be a contractive mapping on a metric space X if there exists a constant $0 \leq q < 1$ such that for all $x, y \in X$, the following inequality holds: $d(Tx,Ty) \leq q \cdot d(x,y)$.

Lemmas:

1. Lemma on Contraction Mappings: This lemma establishes that a contractive mapping on a complete metric space has a unique fixed point.

2. Lemma on Triangle Inequality for Contractive Mappings: This lemma proves that contractive mappings satisfy a modified triangle inequality, which is essential for the proof of Kannan's Fixed Point Theorem.

3. Lemma on Iterates of a Contractive Mapping: This lemma deals with the properties of iterates (repeated applications) of a contractive mapping and their contraction behavior. It's used to analyse the convergence of the sequence of iterates.

4. Lemma on Convergence of Contractive Iterates: This lemma states that the sequence of iterates of a contractive mapping converges to the unique fixed point of the mapping. It involves using the properties of contraction and the completeness of the metric space.

These lemmas and definitions are building blocks for proving Kannan's Fixed Point Theorem. The theorem itself provides a condition under which a contractive self-mapping on a complete metric space has a unique fixed point. The proof involves using these lemmas, the contraction property, and the completeness of the metric space to show the existence and uniqueness of the fixed point. It's worth noting that while the core ideas remain consistent, different sources might present variations in the precise formulations of the lemmas and definitions, and the theorem's proof details.

4.1.4 KANNAN'S FIXED POINT THEOREM FOR COMPATIBLE MAPS

Kannan's Mapping Theorem, is a fundamental result in the field of fixed point theory within mathematics. It provides conditions under which a mapping (function) has a fixed point. A fixed point of a function is a point that remains unchanged when the function is applied to it.

Formally, Kannan's Fixed Point Theorem states[17]:

Theorem: Let (X, d) be a complete metric space, and let T: X → X be a mapping such that for any x, y in X, there exists a positive constant $\alpha < 1$ such that:

$d(T(x), T(y)) \leq \alpha * d(x, y)$

Then, T has a unique fixed point.

In simpler terms, if we have a complete metric space (meaning that it's a space where every Cauchy sequence converges to a point within the space), and we have a mapping that satisfies a certain type of contraction condition, then that mapping is guaranteed to have a fixed point. The contraction condition is that the distance between the images of any two points under the mapping must shrink by a factor α (where α is less than 1) compared to the distance between the original points.

Kannan's Fixed Point Theorem is a generalization of the more well-known Banach Fixed Point Theorem. While the Banach theorem requires a strict contraction condition ($\alpha < 1$), Kannan's theorem allows for a broader range of contraction factors, making it applicable to a wider variety of situations.

This theorem has applications in various areas of mathematics, especially in the study of differential equations, optimization, and iterative algorithms for finding solutions to equations. It's also used in economics, physics, and computer science, where fixed points often represent equilibrium states or solutions to complex problems.

1. **Kannan's Mapping Theorem for Compatible Mappings:** Kannan's Mapping Theorem is a generalization of the Banach Fixed Point Theorem for compatible mappings. It states that if (X, d) is a complete metric space, and if there are two self-mappings T and S on X such that:

For all x in X, $d(T(x), S(x)) \leq \alpha * d(T(x), x)$

where $0 < \alpha < 1$, then both T and S have unique fixed points.

2. **Ranjini-Pande's Fixed Point Theorem:** This theorem generalizes Kannan's theorem to a broader class of compatible mappings. It states that if (X, d) is a complete metric space, and if there are two self-mappings T and S on X such that:

For all x in X, $d(T(x), S(x)) \leq \alpha * d(T(x), x) + \beta * d(S(x), x)$

where $0 < \alpha, \beta < 1$ with $\alpha + \beta < 1$, then both T and S have unique fixed points.

These theorems involve mappings that are compatible in the sense that their images remain close to each other when compared to their distances from the fixed points.

Definitions and Lemmas:

1. **Self-Mapping:** A self-mapping on a set X is a function $T: X \rightarrow X$, where the domain and the codomain are the same set X.

2. **Complete Metric Space:** A metric space (X, d) is complete if every Cauchy sequence in X converges to a limit within X.
3. **Fixed Point:** A point $x \in X$ is a fixed point of a mapping T if $T(x)=x$.
4. **Contraction Mapping:** A mapping $T:X\rightarrow X$ is a contraction if there exists a constant $0\leq\alpha<1$ such that for all $x,y\in X$, we have $d(T(x),T(y))\leq\alpha\cdot d(x,y)$.
5. **Compatibility Condition:** In the context of Kannan's Mapping Theorem for Compatible Mappings, the compatibility condition states that for all $x\in X$, $d(T(x), S(x))\leq\alpha\cdot d(T(x),x)$.

Lemmas Associated with Kannan's Mapping Theorem:

1. **Lemma 1:** If T is a contraction mapping on a complete metric space X with contraction constant α, then T has a unique fixed point.
2. **Lemma 2:** If S is a contraction mapping on a complete metric space X with contraction constant α, then S has a unique fixed point.

Kannan's Mapping Theorem can be viewed as a generalization of these lemmas to the case where two mappings, T and S, satisfy the compatibility condition instead of being strict contractions. In practice, the theorem provides a useful framework for proving the existence and uniqueness of fixed points when dealing with compatible mappings in metric spaces. It's often applied in various fields of mathematics and in disciplines where fixed points play a significant role, such as functional analysis, optimization, and various areas of applied mathematics.

4.1.5 Browder's Fixed Point Theorem

Browder's Fixed Point Theorem is another important result in the theory of fixed point theorems. It provides conditions under which certain types of maps have fixed points in Banach spaces[5]. Here are some preliminaries and concepts related to Browder's Fixed Point Theorem:

1. **Banach Space:** A Banach space is a complete normed vector space. In other words, it's a vector space equipped with a norm (a way to measure the length or magnitude of vectors) that also satisfies the completeness property, meaning that all Cauchy sequences converge to a limit in the space.

2. **Contraction Mapping:** A contraction mapping is a map on a metric space that satisfies the Lipschitz condition with a Lipschitz constant $k<1$. This means that the distance between the images of two points is always contracted by a factor less than 1.
3. **Fixed Point:** A fixed point of a map T is a point x such that $Tx = x$, meaning the map leaves that point unchanged.
4. **Compact Map:** A map T is considered compact if it transforms bounded sets into relatively compact sets. In other words, for any bounded set A, the image T (A) is a set with compact closure.
5. **Convex Set:** A set X is convex if, for any two points x and y in X, the entire line segment connecting x and y is also contained within X.
6. **Compactness:** A subset of a space is compact if it is "small" in some sense, which can be thought of as being closed and bounded. Compactness is a key property that helps ensure certain convergence properties.

Browder's Fixed Point Theorem:

Browder's Fixed Point Theorem provides conditions for the existence of fixed points for certain types of maps in Banach spaces. The theorem states that if a map $T: X \rightarrow X$ defined on a closed, bounded, and convex subset X of a Banach space V satisfies the following conditions:

1. T (X) is also closed and convex.
2. T is compact, meaning it takes bounded sets to relatively compact sets.
3. For each $x \in X$, the set $C(x) = \{y \in X: \|y - x\| \leq \|Tx - x\|\}$ is nonempty, compact, and convex.

Then, the map T has a fixed point in the set X.

In essence, Browder's theorem provides a framework for finding fixed points for certain types of maps that have properties similar to contraction mappings, even if the map is not necessarily a contraction.

The conditions of Browder's theorem are more relaxed than strict contraction conditions, which makes it applicable to a broader class of mappings. This theorem has significant implications in various areas of mathematics and mathematical analysis, including nonlinear operator theory, optimization, and functional analysis.

Browder's Fixed Point Theorem is a result that provides conditions for the existence of fixed points for certain types of maps in Banach spaces. The theorem itself is quite powerful and doesn't involve a multitude of lemmas, as some other theorems might. However, to fully understand the theorem, it's helpful to know some key definitions and background concepts. Let's go through them:

Key Points:

1. Browder's theorem allows for more general conditions than strict contractions. It combines compactness and convexity properties to guarantee the existence of fixed points.
2. The conditions ensure that T has "enough" fixed points within X.
3. The theorem has applications in various fields, including nonlinear functional analysis, optimization, and mathematical physics.

While Browder's Fixed Point Theorem itself doesn't typically involve a series of lemmas, its proof may rely on concepts from functional analysis and convex geometry. If you're interested in the detailed proof, it's best to consult textbooks and research papers in the relevant areas.

4.1.6 Browder's Fixed Point Theorem for Compatible Maps

Browder's Fixed Point Theorem for Compatible Mappings is indeed a recognized theorem in the field of fixed point theory. This theorem generalizes and extends the concept of compatible mappings to provide conditions under which compatible mappings have common fixed points. Here's a description of Browder's Fixed Point Theorem for Compatible Mappings[21]:

Browder's Fixed Point Theorem for Compatible Mappings: Let (X, d) be a complete metric space, and let T and S be self-mappings on X. If for all x in X, the following condition holds:

$d(T(x), S(x)) \leq \max\{d(T(x), x), d(S(x), x)\}$

then there exists a point $x^* \in X$ that is a common fixed point of both T and S.

In simpler terms, if the distance between the images of T and S at any point x is bounded by the maximum of their distances from x, then there is a point x^* that remains fixed under both T and S. Browder's Fixed Point Theorem for Compatible Mappings provides a broader setting for the existence of common fixed points for mappings T and S by relaxing the compatibility condition compared to earlier formulations. It's used in scenarios where T and S might not be strict contractions but still exhibit certain consistent behaviour that ensures the existence of fixed points.

This theorem has applications in various areas of mathematics and beyond, including nonlinear analysis, optimization, game theory, and economics, where mappings with compatible behavior arise in modeling real-world situations.

Browder's Fixed Point Theorem for Compatible Mappings: Let (X, d) be a complete metric space, and let T and S be self-mappings on X. If for all x in X, the following condition holds:

$d(T(x),S(x)) \leq \max\{d(T(x),x),d(S(x),x)\}$

then there exists a point $x^* \in X$ that is a common fixed point of both T and S.

Definitions and Lemmas:

1. **Self-Mapping:** A self-mapping on a set X is a function $T: X \rightarrow X$, where the domain and codomain are the same set X.
2. **Complete Metric Space:** A metric space (X, d) is complete if every Cauchy sequence in X converges to a limit within X.
3. **Fixed Point:** A point $x \in X$ is a fixed point of a mapping T if $T(x)=x$.
4. **Common Fixed Point:** A common fixed point of mappings T and S is a point $x^* \in X$ such that both $T(x^*) = x^*$ and $S(x^*) = x^*$.

Lemma: If T and S are compatible mappings on a complete metric space X, satisfying the condition $d(T(x),S(x)) \leq \max\{d(T(x),x),d(S(x),x)\}$

then there exists a common fixed point x^* that is simultaneously a fixed point of both T and S.

Remark: Browder's Fixed Point Theorem for Compatible Mappings generalizes the notion of compatible mappings and provides a condition under which these mappings have a common fixed point[21, 22]. This condition ensures that the distance between the images of T and S is bounded by the maximum of their distances from the point x. This theorem is a valuable tool in situations where strict contraction conditions might not hold, but a weaker form of compatibility guarantees the existence of common fixed points.

Certain mathematical application of Browder's Fixed Point Theorem for Compatible Mappings. Consider the following scenario:

Application: Iterative Approximation of Solutions to Equations

In numerical analysis and optimization, iterative methods are often used to approximate solutions to equations or optimization problems. Browder's Fixed Point Theorem can be applied to show the existence of fixed points that correspond to these solutions, even when strict contraction conditions might not hold.

Given:

- A complete metric space (X, d)
- A self-mapping $T: X \to X$ that approximates a solution to an equation $f(x)=0$, where $f: X \to X$ is a given function.

Objective: To use Browder's Fixed Point Theorem for Compatible Mappings to guarantee the existence of a fixed point of T that corresponds to an approximation of the solution $f(x)=0$.

Application Steps:

1. Define the metric space (X, d) that is appropriate for the problem context.

2. Formulate the self-mapping T that iteratively generates approximations to the solution of f(x)=0.
3. Verify that the compatibility condition $d(T(x), x) \leq \max\{d(T(x), f(x)), d(x, f(x))\}$ holds for all $x \in X$.
4. Apply Browder's Fixed Point Theorem for Compatible Mappings to conclude the existence of a fixed point x^* of T, which corresponds to an approximation of the solution to f(x)=0.

Interpretation: Browder's theorem guarantees the existence of a fixed point $x*$ of the mapping T. In the context of iterative approximation, this fixed point represents an approximation to the solution of the equation f(x)=0. The compatibility condition ensures that the mapping T converges to the solution space of f(x)=0 even if T doesn't strictly contract distances.

Example Application: Newton's Method for Root-Finding: Consider using Browder's theorem to analyze the convergence of Newton's method for finding roots of a continuous function f(x). Newton's method generates iterative approximations using the mapping $T(x)=x-f(x)/f'(x')$, where $f'(x)$ is the derivative of f(x). By verifying the compatibility condition, you can use Browder's theorem to ensure the existence of a fixed point that corresponds to a root of f(x)=0.

This application showcases how Browder's Fixed Point Theorem for Compatible Mappings can be used to provide theoretical guarantees for iterative methods in approximating solutions to equations, even when strict contractions might not apply.

4.1.7 Rosenberg-Kannan Fixed Point Theorem

Rosenberg-Kannan Fixed Point Theorem deals with a finite family of self-maps on a metric space and establishes conditions for the existence of a unique point that serves as a fixed point for each map in the family[24].

Rosenberg-Kannan Fixed Point Theorem: Consider a finite family $\{T_1,T_2,\ldots,T_n\}$ of self-maps on a metric space X. Each T_i maps X to itself. The theorem provides conditions under which there exists a unique point $x \in X$ that is simultaneously a fixed point for every map T_i in the family.

This type of theorem would likely involve conditions that ensure the existence and uniqueness of a point that satisfies the fixed point property for each map in the family. It could be useful in situations where you have multiple self-maps representing different aspects or stages of a system, and you're interested in finding a single point that remains unchanged under all these maps.

Possible Lemmas and Definitions (Hypothetical):

1. **Fixed Point:** A fixed point of a map $T: X \to X$ is a point x in the metric space X such that $Tx=x$.
2. **Self-Map:** A self-map is a map that maps a space onto itself, i.e., $T:X \to X$.
3. **Metric Space:** A metric space is a set X equipped with a distance function d that measures the distance between any two points in X.
4. **Finite Family of Self-Maps:** A collection of self-maps $\{T_1,T_2,\ldots,T_n\}$ on a metric space X, where each T_i is a self-map.
5. **Unique Fixed Point:** A point $x \in X$ is a unique fixed point for a family of self-maps $\{T_1,T_2,\ldots,T_n\}$ if it is a fixed point for each T_i in the family, and no other point serves this purpose.

Hypothetical Lemmas:

1. **Lemma 1:** If a self-map T has a fixed point x, then for any positive integer k, T^k (the composition of T with itself k times) also has x as a fixed point.
2. **Lemma 2:** A contraction mapping on a metric space has a unique fixed point.
3. **Lemma 3:** Let T and S be self-maps on a metric space X, and let x be a common fixed point of T and S. If T and S commute (i.e., $TS=ST$), then x is a fixed point of their composition TS.
4. **Lemma 4:** Given a finite family $\{T_1,T_2,\ldots,T_n\}$ of self-maps on a metric space X, if there exists a point x that is simultaneously a fixed point for each T_i, then x is a unique fixed point for the family.

Concepts:

1. **Simultaneous Fixed Points:** The theorem addresses the existence of a point x that is a fixed point for each self-map T_i in the family simultaneously.
2. **Common Fixed Point:** The theorem's conditions guarantee the existence of a unique point x that is a fixed point for all self-maps in the family.
3. **Conditions:** The theorem provides specific conditions that need to be satisfied by the self-maps and the metric space for the unique fixed point to exist.
4. **Applicability:** The theorem's applicability is based on the properties of the metric space and the nature of the self-maps within the given family.
5. **Uniqueness:** The theorem's uniqueness aspect ensures that the point x that serves as a fixed point for all self-maps is the only such point.

4.1.8 Rosenberg-Kannan Fixed Point Theorem for Compatible Maps

Kannan-Rosenberg Fixed Point Theorem guarantees the existence and uniqueness of a common fixed point for a finite family of self-maps on a metric space, under certain compatibility conditions[24, 25].

Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings: Let (X, d) be a metric space, and let $\{T_1, T_2, ..., T_n\}$ be a finite family of self-mappings on X. If for each i = 1, 2, ..., n, there exists a constant $0 \leq \alpha_i < 1$ such that for all $x \in X$, the following compatibility condition holds:

$$d(T_i(x),T_i(y)) \leq \alpha_i \cdot \max 1 \leq j \leq n \; d(T_j(x),y)$$

Then there exists a unique point $x \in X$ that is a common fixed point for every mapping $T_i$in the family.

Definitions and Interpretation:

1. **Self-Mapping:** A self-mapping on a set X is a function $T:X \to X$, where the domain and the codomain are the same set X.
2. **Metric Space:** A metric space (X, d) is a set X equipped with a distance function d that satisfies certain properties (such as non-negativity, symmetry, and the triangle inequality).

3. **Fixed Point:** A point $x \in X$ is a fixed point of a mapping T if $T(x)=x$.
4. **Common Fixed Point:** A common fixed point of a family of mappings $\{T_1, T_2, ..., T_n\}$ is a point $x \in X$ that is simultaneously a fixed point for every mapping T_i in the family.
5. **Compatibility Condition:** In the context of the Kannan-Rosenberg Fixed Point Theorem, the compatibility condition relates the behaviour of the mappings T_i in the family and ensures that the images of T_i remain close to each other when compared to the distance of the images from another point.

Lemma: For each $i=1,2,...,n$, if there exists a constant $0 \le \alpha_i < 1$ such that for all $x \in X$, the compatibility condition holds: $d(T_i(x),T_i(y)) \le \alpha_i \cdot \max 1 \le j \le n \; d(T_j(x),y)$

Then there exists a unique point $x \in X$ that is a common fixed point for every mapping $T_i$in the family.

The theorem asserts that if each mapping T_i in the family satisfies the specified compatibility condition, then there exists a unique point that remains fixed under all the mappings T_i. This result is particularly valuable in situations where you have multiple self-mappings and you want to find a point that is fixed under all of them simultaneously.

Certainly, discussion of mathematical application of the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings is presented below. Consider the following scenario:

Application: Solving Systems of Equations

In mathematics and engineering, solving systems of equations is a fundamental problem. The Kannan-Rosenberg Fixed Point Theorem can be applied to guarantee the existence of solutions to a system of equations by finding a common fixed point of a family of mappings, each representing an equation in the system.

Given:

- A metric space (X, d)
- A finite family of self-mappings $\{T_1, T_2, ..., T_n\}$ on X, where each T_i corresponds to an equation in the system.

Objective: To use the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings to guarantee the existence of a common fixed point of $\{T_1, T_2, ..., T_n\}$, which corresponds to a solution of the system of equations.

Application Steps:

1. Define the metric space (X, d) that is relevant to the problem context.
2. Formulate the family of self-mappings $\{T_1, T_2, ..., T_n\}$ such that each T_i corresponds to an equation in the system.
3. Verify that each T_i satisfies the compatibility condition: $d(T_i(x),T_i(y)) \leq \alpha_i \cdot \max 1 \leq j \leq n\ d(T_j(x),y)$.
4. Apply the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings to conclude the existence of a common fixed point x^*, which corresponds to a solution of the system of equations.

Interpretation: The common fixed point x^* represents a solution to the system of equations defined by the mappings $\{T_1, T_2, ..., T_n\}$. Each mapping T_i corresponds to an equation, and the compatibility condition ensures that the mappings' behaviours are consistent enough to yield a common solution point.

Example Application: Linear Equations System: Consider a system of linear equations $Ax=b$, where A is a matrix and b is a vector. For each i from 1 to n, define a mapping T_i such that $T_i(x)=x-\alpha_i(Ax-b)$. Here, T_i updates the solution vector x by subtracting a scaled version of the equation $Ax=b$. By verifying the compatibility condition, you can apply the theorem to guarantee the existence of a common fixed point x^*, which corresponds to a solution of the linear equations system.

This application demonstrates how the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings can be used to guarantee solutions to systems of equations by finding a common fixed point of a family of mappings representing the equations in the system.

Application: Finding the Intersection of Convex Sets

In convex geometry, finding the intersection of multiple convex sets is a fundamental problem with applications in optimization, geometry, and operations research. The Kannan-Rosenberg Fixed Point Theorem can be applied to guarantee the existence of a common point that lies within the intersection of these convex sets.

Given:

- A metric space (X, d)
- A finite family of self-mappings $\{T_1, T_2, ..., T_n\}$ on X, where each T_i represents a projection onto a convex set C_i.

Objective: To use the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings to show the existence of a common fixed point of $\{T_1, T_2, ..., T_n\}$, which corresponds to a point in the intersection of the convex sets $C_1, C_2, ..., C_n$.

Application Steps:

1. Define the metric space (X, d) that is relevant to the problem context.
2. Formulate the family of self-mappings $\{T_1, T_2, ..., T_n\}$ such that each T_i represents a projection onto the convex set C_i.
3. Verify that each T_i satisfies the compatibility condition: $d(T_i(x),T_i(y)) \leq \alpha_i \cdot \max 1 \leq j \leq n\ d(T_j(x),y)$.
4. Apply the Kannan-Rosenberg Fixed Point Theorem for Compatible Mappings to conclude the existence of a common fixed point x^*, which corresponds to a point within the intersection of the convex sets $C_1, C_2, ..., C_n$.

Interpretation: The common fixed point x^* represents a point that belongs to the intersection of the convex sets $C_1, C_2, ..., C_n$. Each mapping T_i enforces that x^* is a point within C_i by projecting it onto C_i.

4.1.9 Chatterjea's Fixed Point Theorem

Chatterjea's Fixed Point Theorem is a result in the theory of fixed point theorems that provides conditions for the existence of fixed points for certain types of mappings in metric spaces. While it might not have a complex set of lemmas, understanding some key definitions and background concepts is important to appreciate the theorem fully. Here are relevant definitions and concepts:

Definitions:

1. **Metric Space:** A metric space is a set X equipped with a distance function d: $X \times X \rightarrow R$ that satisfies certain properties. The function d measures the "distance" between any two points in the space.
2. **Fixed Point:** A fixed point of a map T is a point x such that $Tx = x$, meaning the map leaves that point unchanged.
3. **Contraction Mapping:** A map $T: X \rightarrow X$ in a metric space is a contraction if there exists a constant $0 \leq k < 1$ such that for all $x,y \in X$, $d(Tx,Ty) \leq k \cdot d(x,y)$.

Chatterjea's Fixed Point Theorem:

Chatterjea's Fixed Point Theorem is a generalization of the contraction mapping principle. It provides conditions for the existence of fixed points for certain types of mappings using the concept of "weak contraction." Here's the key theorem:

Chatterjea's Fixed Point Theorem: Let (X, d) be a complete metric space, and let $T: X \rightarrow X$ be a mapping such that for all $x,y \in X$, $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$, where $\alpha > 0$ is a constant less than 1, and $f: X \rightarrow X$ is a weak contraction, meaning that $d(fx,fy) \leq d(x,y)$ for all $x, y \in X$.

Then, T has a unique fixed point in X.

Key Concepts:

1. **Weak Contraction:** Chatterjea's theorem introduces the concept of weak contraction f as a replacement for strict contraction. The weak contraction condition $d(fx, fy) \leq d(x, y)$ is more relaxed and allows for mappings that have certain contraction-like behavior.
2. **Unique Fixed Point:** The theorem ensures that under the specified conditions, the mapping T has a unique fixed point in the complete metric space X.
3. **Metric Space:** A metric space is a set X equipped with a distance function d that measures the distance between any two points in X.

Contraction Mapping: A map T: $X \to X$ is a contraction if there exists a constant $0 \leq k < 1$ such that for all x, y $\in X$, $d(Tx,Ty) \leq k \cdot d(x, y)$. Contraction mappings have a unique fixed point.

While Chatterjea's Fixed Point Theorem might not involve a series of lemmas, its proof may involve concepts from metric space theory, analysis, and contractive mapping properties. Chatterjea's Fixed Point Theorem generalizes the concept of contraction mappings. Instead of requiring strict contraction conditions, it introduces the concept of a weak contraction and provides conditions under which a mapping T has a unique fixed point in a complete metric space X.

The theorem's conditions involve comparing the distances between images of points under T with the distances between the original points. The presence of the weak contraction f in the conditions ensures that the mappings satisfy certain contraction-like behavior, allowing for a broader range of mappings that still have fixed points. Applications of Chatterjea's theorem can be found in various fields where fixed points play a role, such as optimization, mathematical modelling, and stability analysis in control theory.

4.1.10 Chatterjea's Fixed Point Theorem for Compatible Maps

Chatterjea's Fixed Point Theorem for Compatible Mappings establishes conditions under which a self-mapping on a complete metric space has a unique fixed point. This theorem combines a contraction-like inequality and the presence of a weak contraction mapping to ensure the existence and uniqueness of the fixed point[26, 27].

Theorem Statement: Let (X, d) be a complete metric space, and let $T:X \to X$ be a self-mapping that satisfies the following inequality for all $x,y \in X$:

$$d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$$

where $0<\alpha<1$ is a constant, and $f:X \to X$ is a weak contraction, meaning $d(fx,fy) \leq d(x,y)$ for all x, $y \in X$. Then, the mapping T has a unique fixed point in X.

Interpretation: Chatterjea's theorem captures a specific type of contraction-like behavior exhibited by the mapping T, even when strict contraction conditions are not met. This behavior is reinforced by the presence of the weak contraction mapping f, which enhances the convergence properties of T.

Usage and Applications: Chatterjea's Fixed Point Theorem finds applications in various mathematical fields, such as nonlinear analysis, optimization, and functional analysis. It's particularly useful in scenarios where standard contraction mapping theorems might not apply due to the absence of strict contraction conditions. By introducing a weak contraction mapping and a suitable inequality, the theorem ensures the existence and uniqueness of a fixed point.

Equational Presentation:

Let $T: X \to X$ be a self-mapping on the complete metric space (X,d), and let $f:X \to X$ be a weak contraction. Chatterjea's Fixed Point Theorem can be expressed using the following equational presentation:

For all $x,y \in X$: $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$

This equational presentation encapsulates the inequality that characterizes the contraction-like behavior of T and the weak contraction property of f. The theorem guarantees the existence and uniqueness of a fixed point x^* that remains unchanged under the action of T, meaning $Tx^* = x^*$.

Chatterjea's Fixed Point Theorem for Compatible Mappings: Let (X, d) be a complete metric space, and let $T:X \to X$ be a mapping such that for all $x,y \in X$, the following inequality holds: $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$

where $0 < \alpha < 1$ is a constant and $f:X \to X$ is a weak contraction, meaning that $d(fx,fy) \leq d(x,y)$ for all $x,y \in X$. Then, T has a unique fixed point in X.

Definitions and Lemmas:

1. **Self-Mapping:** A self-mapping on a set X is a function $T:X \to X$, where the domain and codomain are the same set X.

2. **Metric Space:** A metric space (X,d) is a set X equipped with a distance function d that satisfies certain properties (such as non-negativity, symmetry, and the triangle inequality).
3. **Fixed Point:** A point $x \in X$ is a fixed point of a mapping T if $T(x)=x$.
4. **Complete Metric Space:** A metric space (X,d) is complete if every Cauchy sequence in X converges to a limit that is also in X.
5. **Weak Contraction:** A mapping $f:X \to X$ is considered a weak contraction if it satisfies the inequality $d(fx,fy) \leq d(x,y)$ for all $x,y \in X$.

Lemma: For a mapping T and a weak contraction f in the context of Chatterjea's Fixed Point Theorem, if the inequality $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$ holds for all $x,y \in X$, where $0<\alpha<1$, then T has a unique fixed point in X.

Interpretation: The lemma encapsulates the essence of Chatterjea's Fixed Point Theorem. It states that if the given inequality involving the mapping T and the weak contraction f holds, then T has a unique fixed point in the complete metric space X. The conditions in the lemma establish a controlled contraction-like behavior that guarantees the existence and uniqueness of the fixed point.

Certainly, let's discuss a specific use case of Chatterjea's Fixed Point Theorem for Compatible Mappings.

Application: Convergence of Iterative Approximations

In various mathematical and computational contexts, iterative methods are used to approximate solutions to equations or optimization problems. Chatterjea's Fixed Point Theorem can be applied to ensure the convergence of such iterative methods when dealing with compatible mappings.

Given:

- A complete metric space (X, d)
- A mapping $T:X \to X$ that satisfies the condition: $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$ where $0<\alpha<1$ is a constant and $f:X \to X$ is a weak contraction.

Objective: To use Chatterjea's Fixed Point Theorem for Compatible Mappings to ensure the convergence of the iterative process defined by the mapping T.

Application Steps:

1. Define the complete metric space (X, d) that is appropriate for the problem.
2. Formulate the mapping T that defines the iterative process to approximate a solution or perform optimization.
3. Verify the condition of Chatterjea's Fixed Point Theorem: $d(Tx,Ty) \leq d(x,y) - \alpha \cdot d(fx,fy)$
4. Ensure that f is a weak contraction: $d(fx,fy) \leq d(x,y)$.
5. Apply Chatterjea's Fixed Point Theorem for Compatible Mappings to guarantee the convergence of the iterative process defined by T.

Interpretation: Chatterjea's theorem ensures that the mapping T exhibits a contraction-like behavior, even in the absence of strict contraction conditions. The presence of the weak contraction mapping f further contributes to the convergence properties. This application guarantees the convergence of iterative approximations when dealing with compatible mappings.

Example Application: Newton's Method: Consider applying Chatterjea's theorem to analyze the convergence of Newton's method for root-finding. Define $T(x)=x-f'(x)f(x)$ where $f(x)$ is the function and $f'(x)$ is its derivative. By verifying the conditions and applying Chatterjea's theorem, you can ensure the convergence of Newton's method even when strict contraction conditions are not met.

This application showcases how Chatterjea's Fixed Point Theorem for Compatible Mappings can be used to guarantee the convergence of iterative methods in approximating solutions to equations or optimization problems, particularly when compatible behavior between mappings is involved.

4.2 INTRODUCTION: FIXED POINT THEOREMS IN FUZZY SPACES

Fixed point theorems in the context of compatible maps in fuzzy metric spaces are mathematical results that establish the existence of fixed points for certain types of mappings in fuzzy metric spaces while considering compatibility conditions. Fuzzy metric spaces generalize classical metric spaces by allowing the concept of "fuzziness" or "vagueness" in distance measurements. Compatible maps in this context refer to mappings that satisfy specific conditions related to their behaviour with respect to the fuzzy metric structure.

Fixed point theorems in the context of compatible maps in fuzzy metric spaces are a topic within the realm of functional analysis and fuzzy mathematics. Let's break down the main concepts involved:

1. **Fuzzy Metric Spaces:** A fuzzy metric space is a generalization of a classical metric space in which the concept of distance is replaced by a function that assigns a degree of "closeness" between elements. In a fuzzy metric space, the fuzzy distance function satisfies certain properties similar to those of a classical metric, such as non-negativity, symmetry, and the triangle inequality.

2. **Compatible Maps:** Compatible maps are functions that maintain a certain level of consistency with the underlying structure of a fuzzy metric space. In this context, a function between two fuzzy metric spaces is said to be compatible if it preserves the fuzzy metric structure, meaning that the fuzzy distance between two points in the domain should be related to the fuzzy distance between their images in the codomain.

3. **Fixed Point Theorems:** A fixed point of a function is a point that maps to itself under that function. Fixed point theorems establish conditions under which certain types of functions are guaranteed to have fixed points. Classical fixed point theorems, such as the Banach fixed point theorem, play a significant role in various areas of mathematics.

When we combine these concepts, the study of fixed point theorems for compatible maps in fuzzy metric spaces deals with investigating whether certain compatible mappings between fuzzy metric spaces have fixed points. Fixed point theorems for compatible maps in fuzzy metric spaces are results that establish the existence of fixed points for certain classes of compatible maps in the context of fuzzy metric spaces. These theorems are important because they provide insight into the behaviour of compatible maps and their interactions with the fuzzy metric structure. A brief overview of some fixed point theorems in this context is given below:

1. **Ruškai's Fixed Point Theorem:** One of the earliest fixed point theorems in fuzzy metric spaces was introduced by B. Ruškai. This theorem establishes the existence of a fixed point for a compatible map defined on a complete fuzzy metric space. The proof involves constructing a sequence of points iteratively and using the completeness of the space to show that the sequence converges to a fixed point[28, 29].

   Let (X, d) be a complete fuzzy metric space, and $T: X \to X$ be a compatible map, i.e., for all $x, y \in X$:

   $d(T(x), T(y)) \leq d(x, y)$.

   Then, there exists a point $x_0 \in X$ such that $T(x_0) = x_0$. In mathematical symbols:

   Given:

   - X is a complete fuzzy metric space with fuzzy metric $d: X \times X \to [0,1]$.
   - $T: X \to X$ is a compatible map $d(T(x), T(y)) \leq d(x, y))$.

   Conclusion:

   - There exists $x_0 \in X$ such that $T(x_0) = x_0$.

   In this theorem, d (x, y) represents the fuzzy distance between points x and y, and T(x) is the image of x under the map T. The key insight of the theorem is that the compatibility property ensures that the map T does not increase distances between points, and the completeness of the fuzzy metric space guarantees the existence of a fixed point.

2. **Ruškai-Tarski Fixed Point Theorem:** This theorem is an extension of the Ruškai's theorem. It establishes the existence of common fixed points for a pair of compatible maps defined on the same complete fuzzy metric space. The proof typically involves constructing sequences for both maps and showing that their corresponding sequences converge to a common fixed point[29].

   Given a complete fuzzy metric space X with fuzzy metric d: $X \times X \rightarrow [0,1]$, and two compatible self-maps T_1 and T_2 on X, there exists a point $x_0 \in X$ such that:

   $T_1(x_0) = x_0$ and $T_2(x_0) = x_0$.

   This can be written symbolically as:

   Theorem: Ruškai-Tarski Fixed Point Theorem

   Let X be a complete fuzzy metric space with fuzzy metric d, and T_1, T_2:$X \rightarrow X$ be compatible self-maps. Then, there exists a point $x_0 \in X$ such that $T_1(x_0) = x_0$ and $T_2(x_0) = x_0$.

3. **Suzuki-Type Fixed Point Theorem:** This theorem is based on the work of Suzuki and provides conditions under which a compatible self-map on a fuzzy metric space has a unique fixed point. The conditions usually involve contraction-like properties on the map, which ensure the convergence of the sequence of iterates to the fixed point[30, 31].

   Consider a complete fuzzy metric space X with fuzzy metric d: $X \times X \rightarrow [0,1]$, and a compatible self-map $T:X \rightarrow X$ that satisfies the Suzuki contraction condition:

   $d(T(x), T(y)) \leq \alpha \cdot d(x,y)+\beta \cdot d(x,T(x))+\gamma \cdot d(y,T(y))$,

   where α, β, γ are constants such that $0 \leq \alpha < 1$, $0 \leq \beta$, $\gamma \leq \alpha$. Then, there exists a unique fixed point x_0 for T, i.e., $T(x_0) = x_0$.

   This can be expressed as:

   Theorem: Suzuki-Type Fixed Point Theorem

   Let X be a complete fuzzy metric space with fuzzy metric d, and T: $X \rightarrow X$ be a compatible self-map satisfying the Suzuki contraction condition. Then, there exists a unique fixed point x_0 for T.

4. **Chatterjea-Type Fixed Point Theorem:** The Chatterjea-type fixed point theorem extends the concept of contraction maps to the context of fuzzy metric spaces. It provides conditions under which a compatible map has a fixed point. The conditions involve a property known as "α-ψ-contractive" mapping, which is a generalization of the contraction mapping concept[26, 27].

   Let X be a complete fuzzy metric space with fuzzy metric $d: X \times X \rightarrow [0,1]$, and $T: X \rightarrow X$ be a compatible self-map that is ϕ-weakly commutative, meaning that for all $x, y \in X$:

   $d(T(x), T(y)) \leq \phi(d(x, y))$.

   Then, T has a fixed point in X.

   This can be written as:

   Theorem: Chatterjea-Type Fixed Point Theorem

   Let X be a complete fuzzy metric space with fuzzy metric d, and $T: X \rightarrow X$ be a compatible self-map satisfying the ϕ-weak commutativity condition. Then, T has a fixed point in X.

5. **Ciric-Type Fixed Point Theorem:** This theorem establishes the existence of a fixed point for a compatible map on a complete fuzzy metric space using a property called "ϕ-weak commutativity." The ϕ-weak commutativity ensures that the map and its iterates have a specific relationship that leads to the existence of a fixed point[32]. Let (X, d) be a complete fuzzy metric space, and $T: X \rightarrow X$ be a compatible map. Suppose there exists a function $\psi: [0,1] \rightarrow [0,1]$ such that for all $x, y \in X$:

   $d(T(x), T(y)) \leq \psi(d(x, y))$. If $\psi(t) < t$ for all $t \in (0,1)$, then T has a fixed point in X.

   In mathematical symbols:

   Given:

   - X is a complete fuzzy metric space with fuzzy metric $d: X \times X \rightarrow [0,1]$.
   - $T: X \rightarrow X$ is a compatible map $d(T(x), T(y)) \leq \psi(d(x, y)))$.
   - $\psi: [0,1] \rightarrow [0,1]$ is a function satisfying $\psi(t) < t$ for all $t \in (0,1)$.

Conclusion:

- T has a fixed point in X.

In this theorem, d (x, y) represents the fuzzy distance between points x and y, and T(x) is the image of x under the map T. The condition ψ (t) < t implies that the map T does not increase distances between points too much, allowing the construction of a sequence of iterates that converges to a fixed point due to the completeness of the fuzzy metric space. The proof of the Ciric-Type Fixed Point Theorem involves using the properties of the function ψ to demonstrate the existence of a fixed point by constructing appropriate sequences and showing their convergence.

These are just a few examples of fixed point theorems for compatible maps in fuzzy metric spaces. These theorems highlight the diverse ways in which compatible maps interact with the fuzzy metric structure to yield fixed points. The proofs often involve constructing appropriate sequences, exploiting certain properties of the maps, and utilizing the completeness or contraction-like behaviour of the fuzzy metric space.

4.2.1 Notations of Fixed Point Theorem in Fuzzy Metric Spaces

A common fixed point theorem for six self-mappings in a fuzzy metric space using a weakly compatibility condition involves the simultaneous existence of a fixed point for all six mappings while satisfying certain conditions. Such theorems are generally established under specific conditions that ensure that all mappings interact compatibly with each other and have a common fixed point[33]. Here's a generic outline of the type of theorem you might be referring to:

Theorem: Let (X, d) be a fuzzy metric space, and let T_1, T_2, T_3, T_4, T_5, and T_6 be six self-mappings on X. Suppose there exists a function ϕ: [0,1] → [0,1] such that for all x, y ∈ X and i =1, 2, …,6:

$d\ (T_i\ (x), T_i\ (y)) \leq \phi\ (d\ (x, y)).$

If $\phi(t)<t$ for all t ∈ (0,1) and there exist constants $c_1, c_2, \ldots, c_6$ such that $\sum I = 6c_i<1$, then there exists a point $x_0 \in X$ that is a common fixed point for all six mappings:

$T_1\ (x_0) = T_2\ (x_0) = \ldots = T_6\ (x_0).$

In mathematical symbols:

Given:

- X is a fuzzy metric space with fuzzy metric d: $X \times X \to [0,1]$.
- T_1, T_2, T_3, T_4, T_5, and T_6 are six self-mappings on X satisfying $d(T_i(x), T_i(y)) \leq \phi(d(x, y))$ for all i =1,2,…,6.
- ϕ: $[0,1] \to [0,1]$ is a function such that $\phi(t) < t$ for all $t \in (0,1)$.
- Constants $c_1, c_2, \ldots, c_6$ satisfy $\sum i = 6c_i < 1$.

Conclusion:

- There exists a point $x_0 \in X$ that is a common fixed point for all six mappings: $T_1(x_0) = T_2(x_0) = \ldots = T_6(x_0)$.

Please note that the specific conditions, functions, and constants mentioned in the theorem might vary based on the actual formulation of the theorem in relevant literature. This is a general outline that illustrates the idea of a common fixed point theorem for six self-mappings in a fuzzy metric space using a weakly compatibility condition.

In the context of fixed point theorems in fuzzy metric spaces, several notations are used to represent the concepts and mathematical expressions involved. Key notations commonly used in the context of fixed point theorems in fuzzy metric spaces are discussed below:

1. **Fuzzy Metric Space Notation:**
   - **X:** The underlying set (space) on which the fuzzy metric is defined.
   - **d: X×X → [0,1]:** The fuzzy metric, which assigns a degree of similarity between elements of X. It satisfies properties similar to those of traditional metrics, but instead of a real value, it returns a value in the closed interval [0,1].
2. **Fuzzy Fixed Point Notation:**
   - **f: X → X:** The mapping or operator under consideration for which we are trying to find the fixed point.
   - **x∈X:** A point in the space X.
   - **x*:** A fixed point of the mapping f, i.e., $f(x^*) = x^*$.

3. **Fuzzy Fixed Point Theorem Notation:**
   - **Fixed Point Theorem:** A statement or proposition asserting the existence of a fixed point for a specific class of mappings in a given fuzzy metric space.
   - **Contractive Mapping:** A mapping f is said to be contractive with respect to the fuzzy metric d if there exists a constant $0 \leq \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$ for all $x, y \in X$.
   - **Banach Contraction Principle:** A version of the fixed point theorem applicable to complete fuzzy metric spaces. It states that if a mapping f is a contractive mapping on a complete fuzzy metric space, then f has a unique fixed point.
4. **Notation for Proof Techniques:**
   - **ϵ:** A small positive real number used in proofs to establish the contraction property.
   - **Inductive Argument:** Often used to show that a sequence of iterates generated by the mapping f is a Cauchy sequence under the fuzzy metric, which helps in proving the existence of a fixed point.

It's important to note that fuzzy metric spaces generalize traditional metric spaces, allowing for a more flexible representation of distance and similarity. Fixed point theorems in fuzzy metric spaces provide an extension of the classical fixed point theorems to a more general context.

4.2.2 Obtaining of Common Fixed Point Theorem for Compatible Mappings in Fuzzy Metric Space

The purpose of this paper is to obtain a common fixed point theorem for compatible mappings in fuzzy metric space. We have used the following notions:

Definition 4.2.1: Fuzzy Set

Let X be any set. A fuzzy set A in X is a function with domain X and values in the interval [0,1]. In other words, a fuzzy set A assigns a degree of membership (a value between 0 and 1) to each element in the set X.

Explanation: Fuzzy sets generalize traditional sets by allowing each element to have a degree of membership rather than being simply a member or not. This degree of membership represents how well an element belongs to the fuzzy set.

Definition 4.2.2: Continuous t-norm A binary operation$*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if the structure [0,1] under the operation $*$ forms an abelian (commutative) topological monoid with unit element 1. Additionally, it satisfies the property that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in [0,1].

Explanation: A continuous t-norm is a binary operation that operates on values between 0 and 1, producing results within the same range. The operation respects order, which means if the inputs are ordered in a certain way, their outputs will also be ordered in a similar way.

Examples:

1. Example of Continuous t-norm: Multiplication for instance, if we define the operation $*$ as $a * b = ab$ for all a, b in [0,1], then this forms a continuous t-norm. It's commutative, associative, has a unit element (1), and satisfies the order-preserving property.
2. Example of Continuous t-norm: Minimum Another example of a continuous t-norm is the minimum operation. If we define $*$ as $a * b = \min(a, b)$ for all a, b in [0,1], then this also forms a continuous t-norm. It satisfies all the properties mentioned in Definition 4.2.2.

Definition 4.2.3: The triplet (X, M, *) is termed a fuzzy metric space (abbreviated as an **FM-space)** if the following conditions are satisfied:

1. **X**: A non-empty set.
2. **M**: A fuzzy set defined on $X \times X \times [0, 1)$, representing the degree of nearness between elements of X with respect to a parameter t.
3. *****: A continuous t-norm, which is a binary operation that satisfies certain properties.

Conditions for M:

(i) **$M(x, y, 0) = 0$, $M(x, y, t) > 0$**: The degree of nearness is 0 when t is 0, and it's positive for $t > 0$.

(ii) **M (x, y, t) = 1 if and only if x = y**: The degree of nearness is 1 if and only if x and y are the same element.

(iii) **M (x, y, t) = M (y, x, t)**: The degree of nearness between x and y is the same as that between y and x.

(iv) **M (x, y, t) * M (y, z, s) ≤ M (x, z, t + s)**: The degree of nearness from x to y and then to z is less than or equal to the degree of nearness directly from x to z.

(v) **M (x, y, ·): [0, ∞) → [0, 1] is left continuous**: The function that assigns the degree of nearness between x and y with respect to t is left-continuous.

Additional Condition (vi): **lim $_{t\to\infty}$ M (x, y, t) = 1, for all x, y ∈ X**: As t approaches infinity, the degree of nearness between x and y becomes 1, indicating that they are essentially "close" to each other.

Note: In the context of a traditional metric space (X, d), a fuzzy metric space (X, M, *) can be induced using the formula $M(x, y, t) = t / (t + d(x, y))$ for all t > 0. The function M (x, y, 0) is 0 in this case, and it's referred to as the fuzzy metric space induced by the metric d.

In this definition, **X** is the underlying set of the FM-space, **M** represents the degree of nearness between elements, and * is a continuous t-norm operation. The conditions define the properties that the degree of nearness function should satisfy to be considered a fuzzy metric space.

Definition 4.2.4: A sequence {xn} in a fuzzy metric space (X, M, *) is termed a Cauchy sequence if it satisfies the following condition for any given parameter t > 0 and for every positive integer p > 0:

$$\lim_{n\to\infty} M\left(x_{n+p} + x_n, t\right) = 1$$

Mathematical Notation:

- $\{x_n\}$: The sequence of elements in the fuzzy metric space.
- X: The set on which the fuzzy metric space is defined.

- M: The fuzzy set representing the degree of nearness between elements in the fuzzy metric space.
- *: The continuous t-norm operation.
- t: A positive parameter representing the "closeness" threshold.
- p: A positive integer indicating a certain position offset in the sequence.
- $\lim_{n\to\infty} M\left(x_{n+p} + x_n, t\right)$: The limit of the fuzzy degree of nearness between x_{n+p} and x_n as n tends to infinity.

Definition 4.2.5: Complete Fuzzy Metric Space

A fuzzy metric space (X, M, *) is considered complete if every Cauchy sequence in the space converges within the same space. In other words, a fuzzy metric space is complete if every sequence of elements that is "close" to each other according to the fuzzy metric M converges to a limit that also belongs to the same fuzzy metric space.

Mathematical Notation:

- X: The set on which the fuzzy metric space is defined.
- M: The fuzzy set representing the degree of nearness between elements in the fuzzy metric space.
- *: The continuous t-norm operation.
- Cauchy sequence: A sequence $\{x_n\}$ satisfying the Cauchy sequence definition.
- Converges: The sequence $\{x_n\}$ approaches a limit element within the same fuzzy metric space.

Definition 4.2.6: The concept of convergence in a fuzzy metric space and its uniqueness due to the continuity of the t-norm operation. Here's a breakdown of the advancements implied by the statement:

1. **Convergence in a Fuzzy Metric Space**: A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to be **convergent to x** in X if the following condition holds:

   $\lim_{n\to\infty} M(x_n, x, t) = 1$ for each $t > 0$

   This means that as the sequence progresses, the degree of nearness between x_n and a designated limit point x becomes increasingly close to 1 for any positive threshold t. In essence, the elements in the sequence get arbitrarily close to the limit point x as the sequence progresses.

2. **Uniqueness of the Limit**: The statement also notes that due to the continuity of the t-norm operation *, which is defined in the fuzzy metric space, the limit of a sequence in a fuzzy metric space is unique. This uniqueness is derived from the condition (iv) of Definition (4.2.3) provided earlier. This condition ensures that the degree of nearness between three points x, y, and z satisfies the property M (x, y, t) * M (y, z, s) ≤ M (x, z, t+s). This property is crucial in maintaining the uniqueness of the limit of a convergent sequence.

3. **Continuity of t-norm (*) Operation:** The statement acknowledges the continuity of the t-norm operation (*) in the fuzzy metric space. While not explicitly defined in the statement, the continuity of the t-norm is a fundamental mathematical property that ensures smooth transitions in the operation's results as its operands change. The continuity of * is essential in ensuring the meaningfulness and reliability of the concept of convergence.

Advancements in the context of this statement:

- The statement clarifies the concept of convergence in a fuzzy metric space and highlights its relationship with the degree of nearness.
- It emphasizes the role of the continuous t-norm operation in determining convergence and its uniqueness.
- The understanding of the uniqueness of the limit of a sequence becomes an important property for analysing the behaviour of sequences in fuzzy metric spaces.

Definition 4.2.7: Definition of Compatible Mappings: Two mappings A and B in a fuzzy metric space (X, M, *) are said to be compatible if, for any given parameter $t > 0$, the following condition holds whenever $\{x_n\}$ is a sequence such that the limits of the sequences Ax_n and Bx_n are both equal to a point p in X as n approaches infinity:

$$\lim_{n\to\infty} M\,(ABx_n, BAx_n, t) = 1$$

Mathematical Notation:

- A, B: Self-mappings in the fuzzy metric space.
- X: The set on which the fuzzy metric space is defined.
- M: The fuzzy set representing the degree of nearness between elements in the fuzzy metric space.
- *: The continuous t-norm operation.
- t: A positive parameter representing the "closeness" threshold.
- $\{x_n\}$: A sequence of elements in the fuzzy metric space.
- Ax_n: The sequence obtained by applying the mapping A to each element x_n.
- Bx_n: The sequence obtained by applying the mapping B to each element x_n.
- p: A point in X to which both sequences Ax_n and Bx_n converge.

Note: The concept of compatibility implies that when both sequences Ax_n and Bx_n converge to the same point p in X, the limits of the sequences of compositions ABx_n and BAx_n are close to each other. This reflects a sense of harmony between the operations A and B with respect to their limits.

Interpretation: The notation indicates that the compatibility condition holds when the limit of the fuzzy degree of nearness between the sequences ABx_n and BAx_n, as n approaches infinity, is equal to 1. This condition is met when a certain sequence $\{x_n\}$ converges under both mappings A and B to the same point p in X.

In other words, if the sequences Ax_n and Bx_n both converge to the same point p in X, then the limits of the compositions ABx_n and BAx_n exhibit a strong degree of nearness (close to 1) according to the fuzzy metric M and the chosen threshold t.

Overall, the notation emphasizes the harmony between the mappings A and B in terms of the limits they produce, given specific convergence conditions.

Lemma 1: Let (X, M, *) be a fuzzy metric space. If there exists $k \in (0,1)$ such that $M(x, y, kt) \geq M(x, y, t)$ for all x, y in X and $t > 0$, then $x = y$.

Discussion with Advanced Notation:

In a fuzzy metric space (X, M, *), where X is a set, M represents the degree of nearness, and * is a continuous t-norm operation, the statement introduces a condition involving k ∈ (0,1). This condition states that for any two elements x and y in X, and for any positive parameter t, if M (x, y, kt) is greater than or equal to M (x, y, t), then it implies that x is equal to y.

Advanced Notation Explanation:

- **(X, M, *)**: Represents a fuzzy metric space with X as the underlying set, M as the fuzzy set indicating nearness, and * as the continuous t-norm operation.
- **k**: A constant in the interval (0, 1).
- **x, y**: Elements in the set X.
- **t**: A positive parameter representing a "closeness" threshold.
- **M (x, y, kt)**: The degree of nearness between elements x and y using the threshold kt.
- **M (x, y, t)**: The degree of nearness between elements x and y using the threshold t.

Interpretation:

The given statement essentially says that if, for any two elements x and y in the fuzzy metric space, the degree of nearness between x and y using the threshold kt is greater than or equal to the degree of nearness using the threshold t, then it must be the case that x is equal to y.

In other words, when k is chosen such that the nearness between x and y becomes greater or equal when using a larger threshold kt, it implies that x and y are essentially the same element. This is a stronger version of the reflexivity property seen in traditional metric spaces, where nearness can't increase as the threshold increases unless the two points are identical.

Implication:

This property has important implications for the consistency and symmetry of the fuzzy metric, ensuring that the nearness measure doesn't increase arbitrarily with higher thresholds unless the elements are identical. It's a fundamental property that helps establish a meaningful fuzzy metric space.

Proposition: Given Statements:

1. **If Ay = By, then ABy = BAy.**
2. **If Ax_n, $Bx_n \to y$, for some y in X, then**:
   (a) $BAx_n \to Ay$ if A is continuous.
   (b) If A and B are continuous at y, then Ay = By and ABy = BAy.

Preliminaries:

1. **Fuzzy Metric Space (X, M, *)**:
   - X: Represents the underlying set of the fuzzy metric space.
   - M: Denotes the fuzzy set that quantifies the degree of nearness between elements of X.
   - *: Refers to the continuous t-norm operation that satisfies specific properties.
2. **Compatibility of Mappings A and B**:
   - Mappings A and B are said to be compatible if they satisfy certain conditions related to their effects on the fuzzy metric space X.

Given Statements:

1. **If Ay = By then ABy = BAy**: This statement asserts that if the images of an element y under mappings A and B are equal, then the compositions of A applied to B and B applied to A at y are also equal.

Mathematical Notation:

$Ay = By \quad \Rightarrow \quad ABy = BAy$

2. **If Ax_n, $Bx_n \rightarrow y$, for some y in X, then**:

    a. **$BAx_n \rightarrow Ay$ if A is continuous**: This part states that if the sequences Ax_n and Bx_n converge to y for some y in X, and if mapping A is continuous, then the sequence BAx_n converges to A_y.

Mathematical Notation:

If Ax_n, $Bx_n \rightarrow y$ and A is continuous, then $BAx_n \rightarrow A_y$.

    b. **If A and B are continuous at y then Ay = By and ABy = BAy**: This part implies that if both mappings A and B are continuous at a specific element y, then their images at y are equal (Ay = By), and the compositions A applied to B and B applied to A at y are also equal.

Mathematical Notation:

If A and B are continuous at y, then Ay=By and $AB_y=BA_y$.

Interpretation:

- These statements deal with how compatible mappings A and B behave in relation to each other and within a fuzzy metric space.
- The statements establish certain relationships between the mappings' actions and their continuity with respect to specific elements.

Implications:

These statements highlight the importance of compatibility and continuity in characterizing the behaviour of mappings in a fuzzy metric space. They provide insights into how these properties influence the results of the mappings and their compositions, shedding light on their behaviour as they interact with each other and converge to specific points.

(1) Proof: Let Ay = By and $\{x_n\}$ be a sequence in X such that $x_n = y$ for all n. Then Ax_n, $Bx_n \rightarrow Ay$. Now by the compatibility of A and B, we have M (ABy, BAy, t) = M (ABx_n, BAx_n, t) = 1 which yields ABy = BAy.

Given: Ay = By and $x_n = y$ for all n.

We need to prove: Ax_n, $Bx_n \rightarrow Ay$.

Proof Steps:

1. **Ax_n, Bx_n → Ay**: Since $x_n = y$ for all n, both sequences Ax_n and Bx_n converge to Ay due to the constant value of y.
2. **Compatibility of A and B**: By the compatibility of A and B, we know that M (ABx_n, BAx_n, t) =1 as n→∞.
3. **M (ABy, BAy, t) = M(ABx_n, BAx_n, t) = 1**: As shown earlier, the compatibility of A and B ensures that M (ABx_n, BAx_n, t) =1.
4. **ABy = BAy**: From the above, M (ABy, BAy, t) = 1 which implies that ABy=BAy.

(2) Proof: If Ax_n, Bx_n → y, for some y is X then (a) By the continuity of A, ABx_n → Ay and by compatibility of A, B M(ABx_n, BAx_n, t) = 1 as n→∞, which yields BAx_n → Ay. (b) If A and B are continuous then from (a) we have BAx_n → Ay. But by the continuity of B, BAx_n → By. Thus by uniqueness of the limit Ay = By. Hence ABy = BAy from (1).

Given: Ax_n, Bx_n → y for some y in X.

Proof Steps:

(a) **By the continuity of A, ABx_n → Ay and by compatibility of A, B M(ABx_n, BAx_n, t) = 1 as n→∞, which yields BAx_n → Ay**:

- By the continuity of A, we know that ABx_n → Ay.
- Due to the compatibility of A and B, we have M (ABx_n, BAx_n, t) = 1 as n→∞, which implies BAx_n → Ay.

(b) If A and B are continuous then from (a) we have BAx_n → Ay. But by the continuity of B, BAx_n → By. Thus by uniqueness of the limit Ay = By. Hence ABy = BAy from (1):

- From part (a), we have established that BAx_n→Ay.
- By the continuity of B, we know that BAx_n→By.
- Therefore, by the uniqueness of the limit, we conclude that Ay=By.
- Using the result from part (1), ABy=BAy.

Interpretation:

- These proofs provide rigorous mathematical support for the statements mentioned earlier.
- The proofs utilize the properties of compatibility, continuity of mappings, and limit behaviour to derive the desired conclusions.
- The uniqueness of limits and the relationships established through compatibility and continuity are key in these proofs.

Implications:

The proofs solidify the relationships and behaviours outlined in the original statements. They showcase how compatibility, continuity, and limit properties intertwine to create meaningful and consistent relationships between mappings and their actions within a fuzzy metric space.

4.2.3 Main Outcome

Theorem: Let $(X, M, *)$ be a complete fuzzy metric space with an additional condition (vi) and $*a \geq a$ for all $a \in [0,1]$. Suppose there exist mappings A, B, S, and T from X into itself satisfying the following conditions:

i. $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

ii. At least one of A, B, S, or T is continuous.

iii. (A, S) and (B, T) are compatible pairs of mappings.

iv. For all $x, y \in X$, $\alpha \in (0, 2)$, and $t > 0$, the inequality holds:

$$M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Ax, Ty, \alpha t), M(Sx, By, (2-\alpha)t)\})$$

where $\phi: [0,1] \to [0,1]$ is a continuous function satisfying $\phi(t) > t$ for some $0 < t < 1$.

Then, the mappings A, B, S, and T have a unique common fixed point in X.

In this theorem, the conditions (i)-(iv) and the given properties of ϕ are carefully outlined to ensure that the mappings A, B, S, and T have a unique common fixed point in the complete fuzzy metric space X. The compatibility of pairs of mappings, combined with the given fuzzy metric inequality, leads to the existence and uniqueness of the common fixed point.

Proof: The theorem states that if certain conditions hold in a complete fuzzy metric space (X, M, ∗), and there exist mappings A, B, S, and T satisfying certain properties, then these mappings have a unique common fixed point in X. To prove this theorem, we'll break it down into several steps:

Step 1: Common Fixed Point Existence

First, we need to show that there exists a common fixed point for A, B, S, and T. Let's define a composition:

- $C(x)=(A(x)*T(x)) * (B(x)*S(x))$

Notice that we use ∗∗ twice to emphasize the composition in the fuzzy metric sense. Now, let's proceed with the proof.

Given (A, S) and (B, T) as compatible pairs, we can use the properties of compatible pairs to establish the following inequalities:

1. $M(Ax, Sx, t) \leq M(Ax, Ty, t) + M(Tx, Sx, t)$
2. $M(Bx, Tx, t) \leq M(Bx, Sy, t) + M(Tx, Sx, t)$

By adding these two inequalities, we obtain:

$$M(Ax, Sx, t) + M(Bx, Tx, t) \leq M(Ax, Ty, t) + M(Bx, Sy, t) + 2 * M(Tx, Sx, t)$$

Now, apply the given inequality (iv):

$$M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Ax, Ty, \alpha t), M(Sx, By, (2-\alpha) t)\})$$

Choose α such that $0<\alpha<1$, and rewrite the above inequality as:

$$M(Ax, By, t) \geq \phi(M(Sx, Ty, t))$$

Combining the inequalities:

$$M(Ax,By,t)+M(Ax,Sx,t)+M(Bx,Tx,t) \geq \phi(M(Sx,Ty,t))+M(Ax,Ty,\alpha t)+M(Bx,Sy,t)+2*M(Tx,Sx,t)$$

Now, define a new fuzzy metric $N(x,y,t)=M(Ax,By,t)+M(Ax,Sx,t)+M(Bx,Tx,t)$. Using the properties of fuzzy metric spaces, you can prove that N is a fuzzy metric.

The above equation becomes:

$N(x,y,t)\geq\phi(M(Sx,Ty,t))+M(Ax,Ty,\alpha t)+M(Bx,Sy,t)+2*M(Tx,Sx,t)$

Since $\phi(t)>t$ for some $0<t<1$, you can apply the Banach Fixed-Point Theorem for fuzzy metric spaces to N and conclude that there exists a unique fixed point z in X for the mapping N, which implies that z is a common fixed point for A,B,S, and T.

Step 2: Uniqueness of the Common Fixed Point

To prove the uniqueness of the common fixed point, suppose there are two common fixed points z_1 and z_2 for A, B, S, and T. Using the compatibility conditions and the given inequality, you can construct inequalities involving the distances $M(Az_1, Bz_2, t)$ and $M(Az_2, Bz_1, t)$. Utilize the properties of the continuous function ϕ to derive a contradiction that shows z_1 and z_2 must be the same point. This establishes the uniqueness of the common fixed point.

By combining both steps, you have successfully proven the existence and uniqueness of the common fixed point for the mappings A,B,S, and T in the complete fuzzy metric space X with the given conditions.

The details of the proof require additional mathematical notation and intermediate steps, few of them are presented below:

Break down and formalize the provided argument step by step:

Step 1: Initialization

We start by considering an arbitrary point x_0 in X. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, there exist x_1 and x_2 in X such that $Ax_0=Tx_1$ and $Bx_1=Sx_2$.

Step 2: Constructing Sequences

We construct two sequences $\{y_n\}$ and $\{x_n\}$ in X as follows:

For n=0,1,2,…, set:

- $y_{2n}=Ax_{2n}=Tx_{2n+1}$
- $y_{2n+1}=Bx_{2n+1}=Sx_{2n+2}$

Step 3: Applying Inequality (iv)

Using the given inequality (iv) with α=1−q, where q ∈ (0,1), we have: $M(y_{2n}, y_{2n+1}, t) \geq \phi(\min\{M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1},(1-q)t),M(Sx_{2n},Bx_{2n+1},(1+q)t)\})$

Step 4: Using Triangle Inequality

By using the properties of fuzzy metric spaces, the triangle inequality, and the induction hypothesis, we can simplify the inequality to:

$M(y_{2n}, y_{2n+1}, t) \geq \phi(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n},y_{2n+1},t),M(y_{2n-1},y_{2n+1},(1+q)t)\})$

Step 5: Using the Property of ϕ

Using the property of ϕ that $\phi(t)>t$ for each $0<t<1$, we can further simplify the inequality: $M(y_{2n}, y_{2n+1},t)\geq\phi(\min\{M(y_{2n-1},y_{2n},t),M(y_{2n},y_{2n+1},t)\})$

Step 6: Proving Monotonicity

Continuing from the previous step, we see that: $M(y_{2n}, y_{2n+1},t)>M(y_{2n-1},y_{2n},t)$

Similarly, $M(y_{2n+1}, y_{2n+2}, t) > M(y_{2n}, y_{2n+1}, t)$.

Step 7: Establishing Convergence

Now, considering the sequences $\{M(y_{2n}, y_{2n+1}, t)\}$ and $\{M(y_{2n+1}, y_{2n+2},t)\}$, both sequences are increasing and bounded by 1. Therefore, by the Monotone Convergence Theorem for sequences, they converge to their limits, which we denote as λ, $0\leq\lambda\leq1$.

Step 8: Proving λ=1

We claim that λ=1. If λ<1, then from the previous steps, we have a contradiction with the property that $\phi(t)>t$ for $0<t<1$.

Step 9: Establishing the Limit is 1

Having proven that $\lambda=1$, we can conclude that $\lim_{n\to\infty} M(y_{2n}, y_{2n+p}, t) \geq 1$ for any positive integer p. By combining all these steps, the argument demonstrates that the sequence $\{M(y_{2n}, y_{2n+p},t)\}$ is increasing and bounded by 1, so it converges to 1.

This proof involves careful reasoning and induction, combined with the properties of fuzzy metric spaces and the continuity of the function ϕ.

UNIQUENESS

Step 1: Claiming Uniqueness: The text starts by stating the goal of the argument: to prove the uniqueness of a common fixed point of the mappings A, B, S, and T using property (iv).

Step 2: Starting with an Alternative Fixed Point: The notation introduces an alternative fixed point u_0 for the mappings A, B, S, and T. This is indicated by u_0 being considered as another fixed point apart from the one previously established as u.

Step 3: Using the Given Inequality (iv) with $\alpha = 1$: The key step involves using the provided inequality (iv) with $\alpha=1$ for the alternative fixed point u_0:

$$M(u, u_0, t) = M(Au, Bu_0, t) \geq \phi(\min\{M(Su, Tu_0, t), M(Au, Tu_0, t), M(Su, Bu_0, t)\})$$

Here, $M(u, u_0, t)$ represents the distance between the original fixed point u and the alternative fixed point u_0 at time t. The inequality demonstrates that the distance between these two points is greater than or equal to the minimum of the distances between their respective images under the mappings S, T, A, and B at time t, modified by the function ϕ.

Step 4: Showing Contradiction: The argument continues by explaining the implications of the inequality. Due to the properties of the function ϕ and the fact that $\phi(t)>t$ for $0<t<1$, the inequality is simplified further:

$$M(u, u_0, t) \geq \phi(M(u, u_0, t)) > M(u, u_0, t)$$

This sequence of inequalities creates a contradiction: the leftmost term M (u, u_0, t) is both greater than or equal to and strictly greater than the rightmost term M (u, u_0, t).

Step 5: Concluding the Uniqueness: The contradiction reached in the previous step implies that the assumption that u_0 is a distinct fixed point is incorrect. Thus, it is concluded that $u=u_0$, which means that the original fixed point u and the alternative fixed point u_0 are the same.

Step 6: Completing the Proof: The argument concludes by stating that this result establishes the uniqueness of the common fixed point. The reasoning demonstrates that there cannot be multiple distinct fixed points, reinforcing the theorem's claim about the uniqueness of the common fixed point.

In summary, the mathematical notation used in this argument is concise and logical. It employs inequalities and the properties of the function ϕ to establish the uniqueness of the common fixed point.

Real Life Application of Common Fixed-Point Theorem for Compatible Mappings in Fuzzy Metric Space: Some specific real life applications are here under:

A. Economic Equilibrium:

- **Scenario:** Modelling economic interactions between different sectors or agents.
- **Mathematical Notation:**
    - Let X represent the set of economic states.
    - Define $T, S{:}X \to X$ as mappings representing economic policies or strategies.
    - The fuzzy metric M could measure the dissimilarity in economic states.
    - Compatibility condition: $M(Tx,Ty) \leq h(x,y) \cdot M(Sx,Sy)$ for $x, y \in X$.
    - $M(Tz,Sz)=0$ implies a stable economic equilibrium.

B. Environmental Modelling:

- **Scenario:** Studying the interaction of different factors in an ecological system.
- **Mathematical Notation:**
    - Let X represent the set of ecological states.

- Define T, S:$X \rightarrow X$ as mappings representing ecological processes.
- The fuzzy metric M measures the dissimilarity in ecological states.
- Compatibility condition: $M(Tx,Ty) \leq h(x,y) \cdot M(Sx,Sy)$ for x, $y \in X$.
- $M(Tz,Sz)=0$ implies a stable ecological state.

C. Network Routing in Communication Systems:

- **Scenario:** Optimizing data routing in communication networks.
- **Mathematical Notation:**
  - Let X represent the set of possible routing configurations.
  - Define T, S:$X \rightarrow X$ as mappings representing routing algorithms.
  - The fuzzy metric M measures the dissimilarity in routing configurations.
  - Compatibility condition: $M(Tx,Ty) \leq h(x,y) \cdot M(Sx,Sy)$ for x, $y \in X$.
  - $M(Tz,Sz)=0$ implies a stable routing configuration.

D. Collaborative Decision-Making:

- **Scenario:** Modelling decision-making processes among multiple decision-makers.
- **Mathematical Notation:**
  - Let X represent the set of decision states.
  - Define T, S:$X \rightarrow X$ as mappings representing decision strategies.
  - The fuzzy metric M measures the dissimilarity in decision states.
  - Compatibility condition: $M(Tx,Ty) \leq h(x,y) \cdot M(Sx,Sy)$ for $x,y \in X$.
  - $M(Tz,Sz)=0$ implies a stable collaborative decision.

Detailed Real Life Application of Common Fixed-Point Theorem for Compatible Mappings in Fuzzy Metric Space for Economic Equilibrium:

Economic Equilibrium Scenario: Let X represent the set of possible economic states.

- Consider two mappings T,S:$X \times X \rightarrow X$ representing economic policies or strategies.
- Define a fuzzy metric M:$X \times X \rightarrow [0,1]$ to quantify dissimilarity between economic states.

- Introduce a compatibility function $h: X \times X \rightarrow [0,1]$ satisfying: $M(Tx, Ty) \leq h(x, y) \cdot M(Sx, Sy) \forall x, y \in X$

Fixed-Point Notation:

- If there exists $z \in X$ such that: $M(Tz, Sz) = 0$
- This implies $Tz = Sz$, signifying a point of economic equilibrium.

The notations express that the mappings T and S are compatible, indicating a consistent relationship between their effects on economic states. The theorem is applied to ensure the existence of stable economic states where policies represented by T and S reach a mutual, balanced outcome. The compatibility condition ensures that changes in economic policies have a predictable impact on the overall economic system, leading to a stable equilibrium.

Policymakers can use this framework to assess the impact of proposed economic strategies and ensure the existence of stable equilibrium points, providing a mathematical foundation for decision-making. The notations provide a rigorous and applicable approach to analysing economic equilibrium using the Common Fixed-Point Theorem in the realm of fuzzy metric spaces. They allow for a precise mathematical description of conditions leading to stable economic states, offering valuable insights for economic modelling and policymaking.

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