CHAPTER – III

FUZZY METRIC SPACE

CONTENTS

3.1 INTRODUCTION AND PRELIMINARIES OF COMPLETE MULTIPLICATIVE METRIC SPACE

Since its introduction by Banach in 1922[1], the Banach contraction principle has sparked considerable interest in the study of fixed and common fixed point theorems for maps[2]. Over the years, scholars have extended this principle to various spaces, such as quasi-metric, fuzzy metric, 2-metric, cone metric, partial metric, and generalized metric spaces[3]. In 2008, Bashirov proposed the concept of multiplicative metric spaces and delved into multiplicative calculus, culminating in the establishment of its fundamental theorem. Building upon this foundation, in 2012, Florack and Assen explored the application of multiplicative calculus in the analysis of biological images[4].

Definition 3.1:[5] Consider a nonempty set A. A multiplicative metric is a function $d: A\times A\to R+$ that fulfills the following conditions:

1. $d(a,b)\geq 1$ for all $a,b\in A$, with equality $d(a,b)=1$ if and only if $a=b$ (referred to as ($M1$).
2. $d(a,b)=d(b,a)$ for all $a,b\in A$ (denoted as ($M2$)).
3. $d(a,b)\leq d(a,c)\cdot d(c,b)$ for all $a,b,c\in A$ (satisfying the multiplicative triangle inequality) ($M3$). The pair (A,d) forms a multiplicative metric space.

Definition 3.2:[6] Given a multiplicative metric space (A,d), a sequence $\{a_n\}$ in A, and $a\in A$, if for every multiplicative open ball $B(a)=\{b|d(a,b)<\epsilon\}$ with $\epsilon>1$, there exists a natural number $N\in$N such that $n\geq N$ implies $a_n\in B(a)$, then $\{a_n\}$ is termed multiplicative convergent to a, denoted as $a_n\to a$ as $n\to\infty$.

Proposition 3.1:[7] For a multiplicative metric space (A,d), a sequence $\{a_n\}$ in A, and $a\in A$, $a_n\to a$ as $n\to\infty$ if and only if $d(a_n,a)\to 1$ as $n\to\infty$.

Definition 3.3: Let (A,d) be a multiplicative metric space[8] and $\{a_n\}$ be a sequence in A. The sequence $\{a_n\}$ is labelled a multiplicative Cauchy sequence if, for every $\epsilon>1$, there exists a positive integer $N\in$N such that $d\ (a_n, a_m)<\epsilon$ for all $n,m\geq N$.

Proposition 3.2: For a multiplicative metric space[8] (A,d) and a sequence $\{a_n\}$ in A, $\{a_n\}$ is a multiplicative Cauchy sequence if and only if $d(a_n,a_m)\to 1$ as $n,m\to\infty$.

Definition 3.4: A multiplicative metric space[8] (A,d) is declared multiplicative complete if every multiplicative Cauchy sequence in (A,d) is multiplicative convergent in A.

Definition 3.5: Let (A,d_A) and (B,d_B) be two multiplicative metric spaces[9], and f:$A\to B$ be a function. f is termed multiplicative continuous at $a\in A$ if for every $\epsilon>1$, there exists $\delta>1$ such that $f(B_\delta(a))\subset B_\delta(f(a))$.

Proposition 3.3: For multiplicative metric spaces[9] (A,d_A) and (B,d_B), a mapping f:$A\to B$, and any sequence $\{a_n\}$ in A, f is multiplicative continuous at $a\in A$ if and only if $f(a_n)\to f(a)$ for every sequence $\{a_n\}$ with $a_n\to a$ as $n\to\infty$.

Proposition 3.4: Given a multiplicative metric space (A,d_A), sequences $\{a_n\}$ and $\{b_n\}$ in A such that $a_n\to a$ and $b_n\to b$ as $n\to\infty$, where $a,b\in A$, $d(a_n,b_n)\to d(a,b)$ as $n\to\infty$.

Definition 3.6: The self-maps f and q of a set A are called commutative if $fqa=qfa$ for all $a\in A$.

Definition 3.7: Suppose f and q are two self-mappings of a multiplicative metric space (A,d)[10]. The pair (f,q) are called weak commutative mappings if $d(fqa,qfa)\leq d(fa,qa)$ for all $a\in A$.

Definition 3.8: Let (A,d) be a multiplicative metric space, and let f:$A\to A$ be called a multiplicative contraction if there exists a real constant $\lambda\in(0,1)$ such that $d(f(a),f(b))\leq d(a,b)\lambda$ for all $a,b\in A$.

Theorem 3.1: Let (A,d) be a multiplicative metric space, and let f:$A\to A$ be a multiplicative contraction. If (A,d) is complete, then f has a unique fixed point.

Theorem 3.2: Let P,Q,M, and N be self-mappings of a multiplicative metric space A, they satisfy the following conditions:

- $P(A)\subset N(A)$, $Q(A)\subset M(A)$;
- M and P are weak commutative, N and Q are also weak commutative;
- one of P, Q, M, and N is continuous;
- $d(Pa,Qb)\leq\{\max\{d(Ma,Nb),d(Ma,Pa),d(Nb,Qb),d(Pa,Nb),d(Ma,Qb)\}\}$, $\lambda\in(0,1/2)$, for all $a,b\in A$. Then P, Q, M, and N have a unique common fixed point.

Definition 3.9: The self-maps f and q of a multiplicative metric space[11] (A,d) are said to be compatible if $\lim_{m\to\infty} d(fq(a_m),qf(a_m))=1$, whenever $\{a_m\}$ is a sequence in A such that $\lim_{m\to\infty} fa_m=\lim_{m\to\infty} qa_m=t$, for some $t\in A$.

Definition 3.10: Suppose that f and q are two self-maps of a multiplicative metric space (A, d). The pair (f, q) are called weakly compatible mappings if $f(a)=q(a)$ for $a\in A$ implies $fq(a)=qf(a)$. That is, $d(fa,qa)=1$ implies $d(fq(a),qf(a))=1$.

Remark 3.1: Commutative mappings must be weak commutative mappings, weak commutative mappings must be compatible, compatible mappings must be weakly compatible, but the converse is not true.

Example 3.1: Let A=R and (A,d) be a multiplicative metric space defined by $d(a,b)=e^{|a-b|}$ for all a,b in A. Let f and q be two self-mappings defined by $f(a)=a^3$ and $q(a)=2-a$. Then $d(f(a_m),q(a_m))= e^{|a_m-1|.|a_{2m}+a_m+2|} \to 1$ if $a_m\to 1$.

iff $a_m\to$d $(fqa_m, qfa_m)=e^{6|a_m-1|^2} = 1$ if $a_m \to 1$

Thus f and q are compatible. Note that $d(fq(0),qf(0))=d(8,2)=e^6>e^2=d(0,0)=d(f(0),q(0))$, so the pair (f,q) is not weakly commuting.

Example 3.2: Let $A=[0,+\infty)$, (A,d) be a multiplicative metric space defined by $d(a,b)=e^{|a-b|}$ for all a,b in A. Let f and q be two self-mappings defined by:

$\text{fa} = \{a, \text{if } 0 \le a < 2, 2, \text{if } a = 24, \text{if } 2 < a < +\infty$

$\text{ga} = \{4 - a, \text{if } 0 \le a < 2, 2, \text{if } a = 27, \text{if } 2 < a < +\infty$

By the definition of the mappings of f and q, only for $a=2$, $fa=qa=2$, at this time $fqa=qfa=2$, so we see the pair (f,q) is weakly compatible. For $a_m=2-1/m\in(0,2)$, from the definition of the mappings of f and q we have $f(a_m)=q(a_m)=2$, but $d(fq(a_m),qf(a_m))=e^{a_m}=e^2\neq 1$, so the pair (f,q) is not compatible.

Let $\emptyset$ denote the set of functions $\varphi:[1,\infty]^5\to[0,\infty)$ satisfying:

- φ is non-decreasing and continuous in each coordinate variable;
- for ≥1,for $t\ge 1$, $\psi(t)=\max\{\varphi(t,t,t,1,t),\varphi(t,t,t,t,1),\varphi(t,1,1,t,t),\varphi(1,t,1,t,1),\varphi(1,1,t,1,t)\}\le t$. From now on, unless otherwise stated, we choose $\varphi \in \emptyset$.

THEOREM 3.2: *Let* (A, d) *be a complete multiplicative metric space,* P, Q, M, *and* N *be four mappings of* A *into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that* $P(X) \subset N(X)$, $Q(X) \subset M(X)$, *and*

$$d(Pa, Qb) \leq \varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb) \quad ..(3.1)$$

for all $a, b \in A$. *Assume one of the following conditions is satisfied*:

a. *Either* M *or* P *is continuous*, *the pair* (P, M) *is compatible and the pair* (Q, N) *is weakly compatible*.

b. *Either* N *or* Q *is continuous*, *the pair* (Q, N) *is compatible and the pair* (P, M) *is weakly compatible*.

Then P, Q, M, *and* N *have a unique common fixed point*.

Proof : Let $a_0 \in A$. Since $P(A) \subset N(A)$ and $Q(A) \subset M(A)$, there exist $a_1, a_2 \in A$ such that $y_0 = Pa_0 = Na_0$ and $y_1 = Qa_1 = Ma_1$. By induction, there exist sequences $\{a_n\}$ and $\{y_n\}$ in A such that

$$Y_{2n} = Pa_{2n} = Na_{2n+1}, y_{2n+1} = Qa_{2n+1} = Ma_{2n+2} \quad ..(3.2)$$

for all $n = 0, 1, 2,...$.

Next, we prove that $\{y_n\}$ is a multiplicative Cauchy sequence in A. In fact, $\forall n \in$ N, from (3.1), (3.2), and the property of ψ we have

$$d(y_{2n}, y_{2n+1}) = d(Pa_{2n}, Qa_{2n+1})$$

$$\leq \varphi\, d^{\lambda}(Ma_{2n}, Na_{2n+1}), d^{\lambda}(Ma_{2n}, Pa_{2n}), d^{\lambda}(Na_{2n+1}, Qa_{2n+1}),$$

$$d^{\lambda}(Pa_{2n}, Na_{2n+1}), d^{\lambda}(Ma_{2n}, Qa_{2n+1})$$

$$= \varphi\, d^{\lambda}(y_{2n-1}, y_{2n}), d^{\lambda}(y_{2n-1}, y_{2n}), d^{\lambda}(y_{2n}, y_{2n+1}), d^{\lambda}(y_{2n}, y_{2n}), d^{\lambda}(y_{2n-2}, y_{2n+1})$$

$$\leq \varphi\;\; d^{\lambda}(y_{2n-1}, y_{2n}), d^{\lambda}(y_{2n-1}, y_{2n}), d^{\lambda}(y_{2n}, y_{2n+1}), 1,$$

$$d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1})$$

$$\leq \varphi\, d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1}), d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1}),$$
$$d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1}), 1, d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1})$$

$\leq \psi\ d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1})$

$\leq d^{\lambda}(y_{2n-1}, y_{2n}) \cdot d^{\lambda}(y_{2n}, y_{2n+1})$.

This implies that

$$d(y_{2n}, y_{2n+1}) \leq d^{\lambda/1-\lambda}(y_{2n-1}, y_{2n}) = d^{h}(y_{2n-1}, y_{2n}). \quad ..(3.3)$$

$h = \lambda/1- \lambda \in (0, 1)$.

Similarly, using (3.1), (3.2), and the property of ψ, we have

$d(y_{2n+1}, y_{2n+1}) = d(Qa_{2n+1}, Pa_{2n+1}) = d(Pa_{2n+1}, Qa_{2n+1})$

$\leq \varphi\ d^{\lambda}(Ma_{2n+1}, Na_{2n+1}), d^{\lambda}(Ma_{2n+1}, Pa_{2n+1}), d^{\lambda}(Na_{2n+1}, Qa_{2n+1}), d^{\lambda}(Pa_{2n+1}, Na_{2n+1}), d^{\lambda}(Ma_{2n+2}, Qa_{2n+1})$

$= \varphi\ d^{\lambda}(y_{2n+2}, y_{2n}), d^{\lambda}(y_{2n+1}, y_{2n+1}), d^{\lambda}(y_{2n}, y_{2n+1}), d^{\lambda}(y_{2n+2}, y_{2n}), d^{\lambda}(y_{2n+1}, y_{2n+1})$

$\leq \varphi\ d^{\lambda}(y_{2n}, y_{2n+1}), d^{\lambda}(y_{2n+1}, y_{2n+1}), d^{\lambda}(y_{2n}, y_{2n+2}), d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+1}, y_{2n+2}), 1$

$\leq \varphi\ d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+1}, y_{2n+2}), d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+2}, y_{2n+1}), d^{\lambda}(y_{2n}, y_{2n+2}) \cdot d^{\lambda}(y_{2n+2}, y_{2n+1}), d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+1}, y_{2n+2}), 1)$

$\leq \psi\ d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+1}, y_{2n+2})$

$\leq d^{\lambda}(y_{2n}, y_{2n+1}) \cdot d^{\lambda}(y_{2n+1}, y_{2n+2})$.

This implies that

$$d(y_{2n+1}, y_{2n+2}) \leq d^{\lambda/1-\lambda}(y_{2n}, y_{2n+1}) = d\,(y_{2n}, y_{2n+1}) \quad ..(3.4)$$

It follows from (3.3) and (3.4) that, for all $n \in$ N, we have

$$d(y_n, y_{n+1}) \leq d^{h}(y_{n-1}, y_n) \leq d^{h_2}(y_{n-2}, y_{n-1}) \leq \cdots \leq d^{h_n}(y_0, y_1).$$

Therefore, for all $n, m \in$ N, $n < m$, by the multiplicative triangle inequality we obtain

$d(y_n, y_m) \leq d(y_n, y_{n+1}) \cdot d(y_{n+1}, y_{n+2})\ d(y_{m-1}, y_m)$

$\leq d^{h_n}(y_0, y_1) \cdot d^{h_{n+1}}(y_0, y_1)\ d^{h_{m-1}}(y_0, y_1)$

$$\leq d^{\frac{h^n}{1-h}}(y_0, y_1).$$

This implies that $d(y_n, y_m) \to 1$ $(n, m \to \infty)$. Hence $\{y_n\}$ is a multiplicative Cauchy sequence in A. By the completeness of A, there exists $z \in A$ such that $y_n \to z$ $(n \to \infty)$.

Moreover, because

$$\{y_{2n}\} = \{Pa_{2n}\} = \{Na_{2n+1}\} \text{ and } \{y_{2n+1}\} = \{Qa_{2n+1}\} = \{Ma_{2n+2}\}$$

are subsequences of $\{y_n\}$, we obtain

$$Pa_{2n} = Na_{2n+1} = Qa_{2n+1} = Ma_{2n+2} = z. \qquad ..(3.5)$$

Next, we prove z is a common fixed point of P, Q, M, and N under the condition (a).

Case 1: Suppose that M is a continuous, then $\lim_{n\to\infty} MPa_{2n} = \lim_{n\to\infty} M^2a_{2n} = Mz$. Since the pair (P, M) is compatible, from (3.5) we have

$$\lim_{n\to\infty} d\,(PMa_{2n}, MPa_{2n}) = \lim_{n\to\infty} d(PMa_{2n}, Mz) = 1,$$

that is, $\lim_{n\to\infty} PMa_{2n} = Mz$. By using (3.1) and (3.2) we have

$$d(PMa_{2n}, Qa_{2n+1}) \le \varphi\, d^{\lambda}(M^2a_{2n}, Na_{2n+1}), d^{\lambda}(M^2a_{2n}, PMa_{2n}), d^{\lambda}(Na_{2n+1}, Qa_{2n+1}),$$

$$d^{\lambda}(PMa_{2n}, Na_{2n+1}), d^{\lambda}(M^2a_{2n}, Qa_{2n+1}))$$

Taking $n \to\infty$ on the two sides of the above inequality, using (3.5) and the property of ψ, we get

$$\begin{aligned} d(Mz, z) &\le \varphi\, d^{\lambda}(Mz, z), d^{\lambda}(Mz, Mz), d^{\lambda}(z, z), d^{\lambda}(Mz, z), d^{\lambda}(Mz, z) \\ &= \varphi\, d^{\lambda}(Mz, z), 1, 1, d^{\lambda}(Mz, z), d^{\lambda}(Mz, z) \\ &\le \psi\, d^{\lambda}(Mz, z) \\ &\le d^{\lambda}(Mz, z). \end{aligned}$$

This means that $d(Mz, z) = 1$, that is, $Mz = z$. Again applying (3.1) and (3.2), we obtain

$$d(Pz, Qa_{2n+1}) \le \varphi\, d^{\lambda}(Mz, Na_{2n+1}), d^{\lambda}(Mz, Pz), d^{\lambda}(Na_{2n+1}, Qa_{2n+1}),$$

$$d^{\lambda}(Pz, Na_{2n+1}), d^{\lambda}(Mz, Qa_{2n+1}).$$

Letting $n \to\infty$ on both sides in the above inequality, using $Mz = z$, (3.5), and the property of ψ, we can obtain

$$d(Pz, z) \leq \varphi\ d^{\lambda}(z, z), d^{\lambda}(z, Pz), d^{\lambda}(z, z), d^{\lambda}(Pz, z), d^{\lambda}(z, z)$$

$$= \varphi\ 1, d^{\lambda}(Pz, z), 1, d^{\lambda}(Pz, z), 1$$

$$\leq \psi\ d^{\lambda}(Pz, z)$$

$$\leq d^{\lambda}(Pz, z).$$

This implies that $d(Pz, z) = 1$, that is, $Pz = z$.

On the other hand, since $z = P(z) \in P(A) \subset N(A)$, there exists $z^* \in A$ such that $z = Pz = Nz^*$.

By using (3.1), $z = Pz = Mz = Nz^*$, and the property of ψ, we can obtain

$$d(z, Qz^*) = d(Pz, Qz^*)$$

$$\leq \varphi\ d^{\lambda}(Mz, Nz^*), d^{\lambda}(Mz, Pz), d^{\lambda}(Nz^*, Qz^*), d^{\lambda}(Pz, Nz^*), d^{\lambda}(Mz, Qz^*)$$

$$= \varphi\ d^{\lambda}(z, z), d^{\lambda}(z, z), d^{\lambda}(z, Qz^*), d^{\lambda}(z, z), d^{\lambda}(z, Qz^*)$$

$$= \varphi(1, 1, d(z, Qz^*), 1, d^{\lambda}(z, Qz^*))$$

$$\leq \psi\ d^{\lambda}(z, Qz^*)$$

$$\leq d^{\lambda}(z, Qz^*).$$

This implies that $d(z, Qz^*) = 1$, and so $Qz^* = z = Nz^*$. Since the pair (Q, N) is weakly compatible, we have $Qz = QNz^* = NQz^* = Nz$.

Now we prove that $Qz = z$. From (3.1) and the property of ψ, we have

$$d(z, Qz) = d(Pz, Qz)$$

$$\leq \varphi\ d^{\lambda}(Mz, Nz), d^{\lambda}(Mz, Pz), d^{\lambda}(Nz, Qz), d^{\lambda}(Pz, Nz), d^{\lambda}(Mz, Qz)$$

$$= \varphi\ d^{\lambda}(z, Qz), d^{\lambda}(z, z), d^{\lambda}(Qz, Qz), d^{\lambda}(z, Qz), d^{\lambda}(z, Qz)$$

$$= \varphi\ d^{\lambda}(z, Qz), 1, 1, d^{\lambda}(z, Qz), d^{\lambda}(z, Qz)$$

$$\leq \psi\ d^{\lambda}(z, Qz)$$

$$\leq d^{\lambda}(z, Qz).$$

This implies that $d(z, Qz) = 1$, so $z = Qz$.

Therefore, we obtain $z = Tz = Mz = Qz = Nz$, so z is a common fixed point of P, Q, M, and N.

Case 2: Suppose that P is continuous, then $\lim_{n\to\infty} PMa_{2n} = \lim_{n\to\infty} P^2a_{2n} = Pz$. Since the pair (P, M) is compatible, from (3.5) we have

$$\lim_{n\to\infty} d(PMa_{2n}, MPa_{2n}) = \lim_{n\to\infty} d(Pz, MPa_{2n}) = 1,$$

that is, $\lim_{n\to\infty} MPa_{2n} = Pz$. From (3.1) and (3.2) we obtain

$$d(P^2a_{2n}, Qa_{2n+1}) \le \varphi\ d^{\lambda}(MPa_{2n}, Na_{2n+1}), d^{\lambda}(MPa_{2n}, P^2a_{2n}), d^{\lambda}(Na_{2n+1}, Qa_{2n+1}),$$

$$d^{\lambda}(P^2a_{2n}, Na_{2n+1}), d^{\lambda}(MPa_{2n}, Qa_{2n+1})$$

Taking $n \to\infty$ on the two sides of the above inequality, using (3.5) and the property of ψ, we can get

$$d(Pz, z) \le \varphi\ d^{\lambda}(\mathrm{P}z, z), d^{\lambda}(Pz, Pz), d^{\lambda}(z, z), d^{\lambda}(Pz, z), d^{\lambda}(Pz, z)$$

$$= \varphi\ d^{\lambda}(Pz, z), z, z, d^{\lambda}(Pz, z), d^{\lambda}(Pz, z)$$

$$\le \psi\ d^{\lambda}(Pz, z)$$

$$\le d^{\lambda}(Pz, z).$$

This means that $d(Pz, z) = 1$, this is $Pz = z$.

Since $z = Pz \in P(A) \subset N(A)$, there exists $z^* \in A$ such that $z = Pz = Nz^*$. From (3.1) we have

$$d(P^2a_{2n}, Qz^*) \le \varphi\ d^{\lambda}(MPa_{2n}, Nz^*), d^{\lambda}(MPa_{2n}, P^2a_{2n}), d^{\lambda}(Nz^*, Qz^*), d^{\lambda}(P^2a_{2n}, Nz^*),$$

$$d^{\lambda}(\mathrm{MP}a_{2n}, Qz^*).$$

Letting $n \to \infty$, using $z = Pz = Nz^*$ and the property of ψ, we can obtain

$$d(z, Qz^*) \le \varphi\ d^{\lambda}(Pz, Nz^*), d^{\lambda}(Pz, Pz), d^{\lambda}(z, Qz^*), d^{\lambda}(Pz, z), d^{\lambda}(Pz, Qz^*)$$

$$= \varphi\ d^{\lambda}(z, z), d^{\lambda}(z, z), d^{\lambda}(z, Qz^*), d^{\lambda}(z, z), d^{\lambda}(z, Qz^*)$$

$$= \varphi\ 1, 1, d^{\lambda}(z, Qz^*), 1, d^{\lambda}(z, Qz^*)$$

$$\le \psi\ d^{\lambda}(z, Qz^*)$$

$$\le d^{\lambda}(z, Qz^*).$$

This implies that $d(z, Qz^*) = 1$, and so $\mathrm{Q}z^* = z = Nz^*$. Since the pair (Q, N) is weakly compatible, we obtain

$$Qz = QNz^* = NQz^* = Nz.$$

So $Qz = Nz$. By (3.1) and the property of ψ, we have

$$d(Pa_{2n}, Qz) \leq \varphi\, d^{\lambda}(Ma_{2n}, Nz), d^{\lambda}(Ma_{2n}, Pa_{2n}), d^{\lambda}(Nz, Qz), d^{\lambda}(Pa_{2n}, Nz), d^{\lambda}(Ma_{2n}, Qz)\,.$$

Taking $n \to\infty$ on the two sides of the above inequality, using $Nz = Qz$ and the property of ψ, we can get

$$\begin{aligned} d(z, Qz) &\leq \varphi\, d^{\lambda}(z, Nz), d^{\lambda}(z, z), d^{\lambda}(Nz, Qz), d^{\lambda}(z, Nz), d^{\lambda}(z, Qz) \\ &= \varphi\, d^{\lambda}(z, Qz), d^{\lambda}(z, z), d^{\lambda}(Qz, Qz), d^{\lambda}(z, Qz), d^{\lambda}(z, Qz) \\ &= \varphi\, d^{\lambda}(z, Qz), 1, 1, d^{\lambda}(z, Qz), d^{\lambda}(z, Qz) \\ &\leq \psi\, d^{\lambda}(z, Qz) \\ &\leq d^{\lambda}(z, Qz). \end{aligned}$$

This implies that $d(z, Qz)= 1$, so $z = Qz = Nz$.

On the other hand, since $z = Qz \in Q(A) \subset M(A)$, there exists $z^{**} \in A$ such that $z = Qz = Mz^{**}$.

By (3.1), using $Qz = Nz = z$ and the property of ψ, we can obtain

$$\begin{aligned} d(Pz^{**}, z) &= d(Pz^{**}, Qz) \\ &\leq \varphi\, d^{\lambda}(Mz^{**}, Nz), d^{\lambda}(Mz^{**}, Pz^{**}), d^{\lambda}(Nz, Qz), d^{\lambda}(Pz^{**}, Nz), d^{\lambda}(Mz^{**}, Qz) \\ &= \varphi\, d^{\lambda}(z, z), d^{\lambda}(z, Pz^{**}), d^{\lambda}(z, z), d^{\lambda}(Pz^{**}, z), d^{\lambda}(z, z) \\ &= \varphi\, 1, d^{\lambda}(Pz^{**}, z), 1, d^{\lambda}(Pz^{**}, z), 1 \\ &\leq \psi\, d^{\lambda}(Pz^{**}, z) \\ &\leq d^{\lambda}(Pz^{**}, z). \end{aligned}$$

This implies that $d(Pz^{**}, z)= 1$, and so $Pz^{**} = z = Mz^{**}$. Since the pair (P, M) is compatible, $d(Pz, Mz)= d\,PMz^{**}, MPz^{**} = d(z, z)= 1$.

So $Mz = Pz$. Hence $z = Pz = Mz = Qz = Nz$.

Next, we prove that P, Q, M, and N have a unique common fixed point. Suppose that

$w \in A$ is also a common fixed point of P, Q, M and N, then

$$d(z, w) = d(Pz, Qw)$$

$\leq \varphi\ d^{\lambda}(Mz, Nw), d^{\lambda}(Mz, Pz), d^{\lambda}(Nw, Qw), d^{\lambda}(Pz, Nw), d^{\lambda}(Mz, Qw)$

$= \varphi\ d^{\lambda}(z, w), d^{\lambda}(z, z), d^{\lambda}(w, w), d^{\lambda}(z, w), d^{\lambda}(z, w)$

$= \varphi\ d^{\lambda}(z, w), 1, 1, d^{\lambda}(z, w), d^{\lambda}(z, w)$

$\leq \psi\ d^{\lambda}(z, w)$

$\leq d^{\lambda}(z, w)$.

This implies that $d(z, w)= 1$, and so $w = z$. Therefore, z is a unique common fixed point of P, Q, M, and N.

Finally, if condition (b) holds, then the argument is similar to that above, so we delete it. This completes the proof.

Example 3.3 Let $A = [0.2]$, and (A, d) be a multiplicative metric space defined by $d(a, b) = e^{|a-b|}$ for all a, b in A. Let P, Q, M, and N be four self-mappings defined by

$Pa = \frac{5}{4}, \forall a \in [0, 2]$, Qa $=\{ \frac{7}{4}, a \in [0,1]\ \frac{5}{4}, a \in [1,2]\}$

Ma $= \{1, a \in [0,1]\ \frac{7}{4}, a \in [1,2]\ \frac{7}{4}, a = 2\}$

Na $= \{\frac{1}{4}, a \in [0,1]\ \frac{5}{4}, a \in [1,2]\ 1, a = 2\}$

Note that P is multiplicative continuous in A, and Q, M, and N are not multiplicative continuous mappings in A.

i. Clearly we can get $P(A) \subset N(A)$ and $Q(A) \subset M(A)$.

ii. By the definition of the mappings of P and M, only for $\{a_n\} \subset (P, M)$, we have

$\lim_{n\to\infty} Pa_n = \lim_{n\to\infty} Ma_n = t = \frac{5}{4}$

$\lim_{n\to\infty} d(PMa_n, MPa_n) = d(\frac{5}{4}, \frac{5}{4}) = 1$

so we can see the pair (P, M) is compatible.

By the definition of the mappings of Q and N, only for $a \in (1, 2)$, $Qa = Na = \frac{5}{4}$, $QNa = Q(\frac{5}{4}) = \frac{5}{4} = N(\frac{5}{4}) = NQa$, so $QNa = NQa$, thus we can see the pair (Q, N) to be weakly compatible.

Now we prove that the mappings P, Q, M and N satisfy the condition (3.1) of Theorem with $\lambda = \frac{2}{3}$ and $\varphi\,(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2}(t_1 + t_2 + t_3 + t_4 + t_5)$. For this, we consider the following cases:

Case 1. If $a, b \in [0, 1]$, then

$$d(Pa, Qb) = d\left(\frac{5}{4}, \frac{7}{4}\right) = e^{\frac{1}{2}}$$

and

$$\varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)$$

$$= \varphi\, d^{\frac{2}{3}}\left(\frac{5}{4}, \frac{1}{4}\right), d^{\frac{2}{3}}\left(\frac{5}{4}, \frac{5}{4}\right), d^{\frac{2}{3}}\left(\frac{1}{4}, \frac{7}{4}\right), d^{\frac{2}{3}}\left(\frac{5}{4}, \frac{1}{4}\right), d^{\frac{2}{3}}\left(\frac{5}{4}, \frac{7}{4}\right),$$

$$= \varphi(e^{\frac{2}{3}}, 1, e, e^{\frac{2}{3}}, e^{\frac{1}{3}})$$

$$= \frac{1}{5}(e^{\frac{2}{3}}, 1, e, e^{\frac{2}{3}}, e^{\frac{1}{3}})$$

$$= e^{\frac{1}{2}}.\frac{1}{5}(e^{\frac{1}{6}} + e^{-\frac{1}{2}} + e^{\frac{1}{2}} + e^{\frac{1}{6}} + e^{-\frac{1}{6}})$$

$$> e^{\frac{1}{3}}$$

Thus we have

$$d(Pa, Qb) = e^{\frac{1}{2}} < \varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)\,.$$

Case 2. If $a = 2$, $b \in (0, 1]$, then we obtain

$$d(Pa, Qb) = d\left(\frac{5}{4}, \frac{5}{4}\right) = e^{\frac{1}{2}}$$

and

$$\varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)$$

$$= \varphi\, d^{\frac{2}{3}}\left(\frac{7}{4}, \frac{1}{4}\right), d^{\frac{2}{3}}\left(\frac{7}{4}, \frac{5}{4}\right), d^{\frac{2}{3}}\left(\frac{1}{4}, \frac{7}{4}\right), d^{\frac{2}{3}}\left(\frac{5}{4}, \frac{1}{4}\right), d^{\frac{2}{3}}\left(\frac{7}{4}, \frac{7}{4}\right),$$

$$= \varphi(e, e^{\frac{1}{3}}, e, e^{\frac{2}{3}}, 1)$$

$= \frac{1}{5}(e + e^{\frac{1}{3}} + e + e^{\frac{2}{3}} + 1)$

$= e^{\frac{1}{2}}.\frac{1}{5}(e^{\frac{1}{2}} + e^{-\frac{1}{6}} + e^{\frac{1}{2}} + e^{\frac{1}{6}} + e^{-\frac{1}{2}})$

$> e^{\frac{1}{2}}$

$d(Pa, Qb) = e^{\frac{1}{2}} < \varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)$

***Case* 3.** If $a, b \in [1, 2]$, then

$d(Pa, Qb) = d\left(\frac{5}{4}, \frac{5}{4}\right)$

$= 1 \leq \varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)$.

Then in all the above cases, the mappings P, Q, M, and N satisfy the condition (3.1) of Theorem 3.3 with $\lambda = 2/3$ and $\varphi(t_1, t_2, t_3, t_4, t_5) = 1/5(t_1 + t_2 + t_3 + t_4 + t_5)$. So all the conditions of Theorem 3.3 are satisfied. Moreover, 5/4 is the unique common fixed point for all of the mappings P, Q, M, and N.

THEOREM 3.3: *Let (A, d) be a complete multiplicative metric space P, Q, M and N be four mappings of A into itself. Suppose that there exist $\lambda \in (0, \frac{1}{2})$ and $u, v \in Z^{+}$ such that $P(A) \subset N(A)$, $Q(A) \subset M(A)$, and*

$$d(P^{u}a, Q^{v}b) \leq \varphi\, d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, P^{u}a), d^{\lambda}(Nb, Q^{v}b), d^{\lambda}(P^{u}a, Nb), d^{\lambda}(Ma, Q^{v}b) \quad ..(3.6)$$

for all $a, b \in A$. Assume the following conditions are satisfied:

a. *the pairs (P, M) and (Q, N) are commutative mappings;*

b. *one of P, Q, M, and N is continuous.*

Then P, Q, M and N have a unique common fixed point.

Proof: From $P(A) \subset N(A)$, $Q(A) \subset M(A)$ we have

$P^{u}(A) \subset P^{u-1}(A) \subset \cdots \subset P^{2}(A) \subset P(A) \subset N(A)$

and

$Q^{v}(A) \subset Q^{v-2}(A) \subset \cdots \subset Q^{2}(A) \subset Q(A) \subset M(A)$.

Since the pairs (P, M) and (Q, N) are commutative mappings,

$P^u(M) = P^{u-1}(PM) = P^{u-1}(MP) = P^{u-2}(PM)P = P^{u-2}(MP^2) = \cdots = (M)P^u$

and

$Q^v(N) = Q^{v-1}(QN) = Q^{v-1}(NQ) = Q^{v-2}(QN)Q = Q^{v-2}(NQ^2) = \cdots = (N)Q^v$.

That is to say, $P^uM = MP^u$ and $Q^vN = NQ^v$.

It follows from Remark 3.1 that the pairs (P^u, M) and (Q^v, N) are compatible and also weakly compatible. Therefore, by Theorem 3.3, we can find that P^u, Q^v, M, and N have a unique common fixed point z.

In addition, we prove that P, Q, M and N have a unique common fixed point. From (3.6) and the property of ψ we have

$d(Pz, z) = d(P^u(Pz), Q^vz)$

$\leq \varphi\, d^{\lambda}(MPz, Nz), d^{\lambda}(MPz, P^u(Pz)), d^{\lambda}(Mz, Q^vz), d^{\lambda}(P^u(Pz), Nz), d^{\lambda}(MPz, Q^vz)$

$= \varphi\, d^{\lambda}(Pz, z), d^{\lambda}(Pz, Pz), d^{\lambda}(z, z), d^{\lambda}(Pz, z), d^{\lambda}(Pz, z)$

$= \varphi\, d^{\lambda}(Pz, z), 1, 1, d^{\lambda}(Pz, z), d^{\lambda}(Pz, z)$

$\leq \psi\, d^{\lambda}(Pz, z)$

$\leq d^{\lambda}(Pz, z)$.

This implies that $d(Pz, z)= 1$, so $Pz = z$.

On the other hand, we have

$d(z, Qz) = d(P^uz, Q^v(Qz))$

$\leq \varphi\, d^{\lambda}(Mz, NQz), d^{\lambda}(Mz, P^uz), d^{\lambda}(NQz, Q^v(Qz)), d^{\lambda}(P^uz, NQz), d^{\lambda}(Mz, Q^v(Qz))$

$= \varphi\, d^{\lambda}(Pz, z), d^{\lambda}(z, z), d^{\lambda}(Qz, Qz), d^{\lambda}(z, Qz), d^{\lambda}(z, Qz)$

$= \varphi\, d^{\lambda}(Pz, z), 1, 1, d^{\lambda}(z, Qz), d^{\lambda}(z, Qz)$

$\leq \psi\, d^{\lambda}(z, Qz)$

$\leq d^{\lambda}(z, Qz)$.

This implies that $d(z, Qz)= 1$, *i.e.*, $Qz = z$.

Therefore, we obtain $Pz = Qz = Mz = Nz = z$, so z is a common fixed point of P, Q, M and N.

Finally, we prove that P, Q, M, and N have a unique common fixed point. Suppose that $w \in A$ is also a common fixed point of P, Q, M and N, then

$d(z, w) = d(P^u z, Q^v w)$

$\leq \varphi\ d^{\lambda}(Mz, Nw), d^{\lambda}(Mz, P^u z), d^{\lambda}(Nw, Q^v w), d^{\lambda}(P^u z, Nw), d^{\lambda}(Mz, Q^v w)$

$= \varphi\ d^{\lambda}(z, w), d^{\lambda}(z, z), d^{\lambda}(w, w), d^{\lambda}(z, w), d^{\lambda}(z, w)$

$= \varphi\ d^{\lambda}(z, w), 1, 1, d^{\lambda}(z, w), d^{\lambda}(z, w)$

$\leq \psi\ d^{\lambda}(z, w)$

$\leq d^{\lambda}(z, w)$.

This implies that $d(z, w) = 1$, and so $w = z$. Therefore, z is a unique common fixed point of P, Q, M, and N.

Corollary 3.1: *Let* (A, d) *be a complete multiplicative metric space* P, Q, M *and* N *be four mappings of* A *into itself. Suppose that there exists* $\lambda \in (0,1/2)$ *such that* $P(A) \subset N(A)$, $Q(A) \subset M(A)$, *and*

$$d(Pa, Qb) \leq \max d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb) \quad ..(3.7)$$

for all $a, b \in A$. *Assume one of the following conditions is satisfied*:

a. *either* M *or* P *is continuous*, *the pair* (P, M) *is compatible and the pair* (Q, N) *is weakly compatible;*

b. *either* N *or* Q *is continuous*, *the pair* (Q, N) *is compatible and the pair* (P, M) *is weakly compatible*.

Then P, Q, M *and* N *have a unique common fixed point.*

Corollary 3.2: *Let* (A, d) *be a complete multiplicative metric space* P, Q, M *and* N *be four mappings of* A *into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that* $P(A) \subset N(A)$, $Q(A) \subset M(A)$, *and*

$$d(P^u a, Q^v b) \leq \max d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, P^u a), d^{\lambda}(Nb, Q^v b), d^{\lambda}(P^u a, Nb), d^{\lambda}(Ma, Q^v b) \quad ..(3.8)$$

for all $a, b \in A$. *Assume the following conditions are satisfied*:

a. *the pairs* (P, M) *and* (Q, N) *are commutative mappings*;

b. *one of* P, Q , M *and* N *is continuous*.

Then P, Q, M and N have a unique common fixed point.

Corollary 3.3: *Let* (A, d) *be a complete multiplicative metric space* P, Q, M *and* N *be four mappings of* A *into itself. Suppose that there exists* $\lambda \in (0,1/2)$ *such that* $P(A) \subset N(A)$, $Q(A) \subset M(A)$, *and*

$$d(Pa, Qb) \leq e_1 d^{\lambda}(Ma, Nb) + e_2 d^{\lambda}(Ma, Pa) + e_3 d^{\lambda}(Nb, Qb) + e_4 d^{\lambda}(Pa, Nb) + e_5 d^{\lambda}(Ma, Qb) \quad ..(3.9)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0 < e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$.

Assume one of the following conditions is satisfied:

a. *either* M *or* P *is continuous*, *the pair* (P, M) *is compatible and the pair* (Q, N) *is weakly compatible*;

b. *either* N *or* Q *is continuous*, *the pair* (Q, N) *is compatible and the pair* (P, M) *is weakly compatible*.

Then P, Q, M *and* N *have a unique common fixed point*.

Proof: Suppose the condition (3.9) hold. For $a, b, c \in A$, let

$R(a, b, c) = \max d^{\lambda}(Ma, Nb), d^{\lambda}(Ma, Pa), d^{\lambda}(Nb, Qb), d^{\lambda}(Pa, Nb), d^{\lambda}(Ma, Qb)$.

Then

$e_1 d^{\lambda}(Na, Mb) + e_2 d^{\lambda}(Ma, Pa) + e_3 d^{\lambda}(Nb, Qb) + e_4 d^{\lambda}(Pa, Nb) + e_5 d^{\lambda}(Ma, Qb)$

$\leq (e_1 + e_2 + e_3 + e_4 + e_5) R(a, b, c)$

$\leq R(a, b, c)$.

So, if (3.9) holds, then $d(Pa, Qb) \leq R(a, b, c)$ for all $a, b, c \in A$. Then the conclusion of Corollary 3.1 can be obtained from Corollary 3.1 immediately.

Corollary 3.4: *Let* (A, d) *be a complete multiplicative metric space* P, Q, M *and* N *be four mappings of* A *into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that* $P(A) \subset N(A)$, $Q(A) \subset M(A)$ *and*

$$d(P^u a, Q^v b) \leq e_1 d^{\lambda}(Ma, Nb) + e_2 d^{\lambda}(Ma, P^u a) + e_3 d^{\lambda}(Nb, Q^v b) + e_4 d^{\lambda}(P^u a, Nb) + e_5 d^{\lambda}(Ma, Q^v b) \quad ..(3.10)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0 < e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$.

Assume the following conditions are satisfied:

a. *the pairs* (P, M) *and* (Q, N) *are commutative mappings*;

b. *one of* P, Q, M *and* N *is continuous*.

Then P, Q, M *and* N *have a unique common fixed point*.

Proof: It is similar to the proof of Theorem 3.4.

By taking $M = N = I$ (the identity mappings) in Theorems 3.3 and 3.4, and Corollaries 3.1 and 3.2, we have the following results.

Corollary 3.5: *Let* (A, d) *be a complete multiplicative metric space*, *P and Q be two mappings of A into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Pa, Qb) \leq \varphi\, d^{\lambda}(a, b), d^{\lambda}(a, Pa), d^{\lambda}(b, Qb), d^{\lambda}(Pa, b), d^{\lambda}(a, Qb) \qquad ..(3.11)$$

for all $a, b \in A$. *Then P and Q have a unique common fixed point*.

Corollary 3.6: *Let* (A, d) *be a complete multiplicative metric space P and Q be two mappings of A into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^{+}$ *such that*

$$d(P^{u}a, Q^{v}b) \leq \varphi\, d^{\lambda}(a, b), d^{\lambda}(a, P^{u}a), d^{\lambda}(b, Q^{v}b), d^{\lambda}(P^{u}a, b), d^{\lambda}(a, Q^{v}b) \qquad ..(3.12)$$

for all $a, b \in A$. *Then P and Q have a unique common fixed point*.

Corollary 3.7: *Let* (A, d) *be a complete multiplicative metric space P and Q be two mappings of A into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Pa, Qb) \leq \max\, d^{\lambda}(a, b), d^{\lambda}(a, Pa), d^{\lambda}(b, Qb), d^{\lambda}(Pa, b), d^{\lambda}(a, Qb) \qquad ..(3.13)$$

for all $a, b \in A$. *Then P and Q have a unique common fixed point*.

Corollary 3.8: *Let* (A, d) *be a complete multiplicative metric space P and Q be two mappings of A into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^{+}$ *such that*

$$d(P^{u}a, Q^{v}b) \leq \max\, d^{\lambda}(a, b), d^{\lambda}(a, P^{u}a), d^{\lambda}(b, Q^{v}b), d^{\lambda}(P^{u}a, b), d^{\lambda}(a, Q^{v}b) \qquad ..(3.14)$$

for all $a, b \in A$. *Then P and Q have a unique common fixed point*.

Corollary 3.9: *Let* (A, d) *be a complete multiplicative metric space*, *P and Q be two mappings of A into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Pa, Qb) \leq e_1 d^{\lambda}(a, b) + e_2 d^{\lambda}(a, Pa) + e_3 d^{\lambda}(b, Qb) + e_4 d^{\lambda}(Pa, b) + e_5 d^{\lambda}(a, Qb) \qquad ..(3.15)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0< e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$. *Then P and Q have a unique common fixed point*.

Corollary 3.10: *Let* (A, d) *be a complete multiplicative metric space*, *P and Q be two mappings of A into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that*

$$d(P^u a, Q^v b) \leq e_1 d^\lambda(a, b) + e_2 d^\lambda(a, P^u a) + e_3 d^\lambda(b, Q^v b) + e_4 d^\lambda(P^u a, b) + e_5 d^\lambda(a, Q^v b) \quad ..(3.16)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0< e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$. *Then P and Q have a unique common fixed point*.

By taking $P = Q$ in Corollaries 3.5 - 3.10, we have the following results.

Corollary 3.11 *Let* (A, d) *be a complete multiplicative metric space*, *P be a mapping of A into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Qa, Qb) \leq \varphi\, d^\lambda(a, b), d^\lambda(a, Qa), d^\lambda(b, Qb), d^\lambda(Qa, b), d^\lambda(a, Qb) \quad ..(3.17)$$

for all $a, b \in A$. *Then Q have a unique fixed point*.

Corollary 3.12: *Let* (A, d) *be a complete multiplicative metric space*, *Q be a mapping of A into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that*

$$d(Q^u a, Q^v b) \leq \varphi\, d^\lambda(a, b), d^\lambda(a, Q^u a), d^\lambda(b, Q^v b), d^\lambda(Q^u a, b), d^\lambda(a, Q^v b) \quad ..(3.18)$$

for all $a, b \in A$. *Then Q have a unique fixed point*.

Corollary 3.13: *Let* (A, d) *be a complete multiplicative metric space*, *Q be a mapping of A into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Qa, Qb) \leq \max d^\lambda(a, b), d^\lambda(a, Qa), d^\lambda(b, Qb), d^\lambda(Qa, b), d^\lambda(a, Qb) \quad ..(3.19)$$

for all $a, b \in A$. *Then Q has a unique fixed point*.

Corollary 3.14: *Let* (A, d) *be a complete multiplicative metric space*, *Q be a mapping of A into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that*

$$d(Q^u a, Q^v b) \leq \max d^\lambda(a, b), d^\lambda(a, Q^u a), d^\lambda(b, Q^v b), d^\lambda(Q^u a, b), d^\lambda(a, Q^v b) \quad ..(3.20)$$

for all $a, b \in A$. *Then Q has a unique fixed point*.

Corollary 3.15 *Let* (A, d) *be a complete multiplicative metric space*, Q *be a mapping of* A *into itself. Suppose that there exists* $\lambda \in (0, 1/2)$ *such that*

$$d(Qa, Qb) \leq e_1 d^{\lambda}(a, b) + e_2 d^{\lambda}(a, Qa) + e_3 d^{\lambda}(b, Qb) + e_4 d^{\lambda}(Qa, b) + e_5 d^{\lambda}(a, Qb) \quad ..(3.21)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0 < e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$. *Then* Q *has a unique fixed point*.

Corollary 3.16 *Let* (A, d) *be a complete multiplicative metric space*, Q *be a mapping of* A *into itself. Suppose that there exist* $\lambda \in (0, 1/2)$ *and* $u, v \in Z^+$ *such that*

$$d(Q^u a, Q^v b) \leq e_1 d^{\lambda}(a, b) + e_2 d^{\lambda}(a, Q^u a) + e_3 d^{\lambda}(b, Q^v b) + e_4 d^{\lambda}(Q^u a, b) + e_5 d^{\lambda}(a, Q^v b) \quad ..(3.22)$$

for all $a, b \in A$. *Here* $e_1, e_2, e_3, e_4, e_5 \geq 0$ *and* $0 < e_1 + e_2 + e_3 + e_4 + e_5 \leq 1$. *Then* Q *has a unique fixed point*.

3.4 CONCLUSION

The chapter undertook comprehensive exploration of fuzzy metric spaces, a fundamental concept in mathematical analysis. Our exploration began with a meticulous examination of the definitions, properties, and formal mathematical notations linked to fuzzy metric spaces. Through an in-depth investigation of these foundational elements, our goal was to establish a robust foundation for comprehending the distinctive characteristics and applications of this mathematical framework.

The definition of fuzzy metric spaces introduced a nuanced perspective by incorporating fuzzy scalars to redefine distance measures. This departure from conventional metric spaces not only broadens the mathematical framework but also enhances its capability to model uncertainty and vagueness in real-world scenarios. The integration of fuzzy logic in defining fuzzy metric spaces adds a layer of flexibility and adaptability, enabling a more nuanced representation of imprecise information.

❒❒❒

REFERENCES

[1.] S. Abbas, M. Benchohra, J. R. Graef & J. Henderson. Implicit fractional differential and integral equations. In *Implicit Fractional Differential and Integral Equations*. de Gruyter, 2018.

[2.] A. A. N. Abdou. Common fixed point results for compatible-type mappings in multiplicative metric spaces. *J. Nonlinear Sci. Appl*, 2016; *9*, 2244–2257.

[3.] P. Agarwal, M. Jleli, B. Samet, P. Agarwal, M. Jleli, & B. Samet. Banach contraction principle and applications. *Fixed Point Theory in Metric Spaces: Recent Advances and Applications*, 2018; 1–23.

[4.] M. A. Ahmed, & A. M. Zidan. Fixed point theorems in a generalized type of multiplicative metric spaces. *International Journal of Advances in Mathematics*, 2016; *1*(1), 49–54.

[5.] K. Kumar, N. Sharma, R. Jha, A. Mishra, & M. Kumar. Weak Compatibility and Related Fixed Point Theorem for Six Maps in Multiplicative Metric Space. *Turkish Journal of Analysis and Number Theory*, 2016; *4*(2), 39–43.

[6.] T. Nazir, & S. Silvestrov. Some Periodic Point and Fixed Point Results in Multiplicative Metric Spaces. In *Stochastic Processes, Statistical Methods, and Engineering Mathematics: SPAS 2019, Västerås, Sweden, September 30–October 2,* 2023; (pp. 367–382). Springer.

[7.] M. Özavşar. Fixed points of multiplicative contraction mappings on multiplicative metric spaces. *Journal of Engineering Technology and Applied Sciences*, 2017a; *2*(2), 65–79.

[8.] M. Özavşar. Fixed points of multiplicative contraction mappings on multiplicative metric spaces. *Journal of Engineering Technology and Applied Sciences*, 2017b; *2*(2), 65–79.

[9.] A. Ozyapici & B. Bilgehan. Finite product representation via multiplicative calculus and its applications to exponential signal processing. *Numerical Algorithms*, 2016; *71*, 475–489.

[10.] T. Rasham & M. De La Sen. A novel study for hybrid pair of multivalued dominated mappings in b-multiplicative metric space with applications. *Journal of Inequalities and Applications*, *2022* (1), 107.

[11.] B. Vijayabaskerreddy & V. Srinivas. Fixed point results on multiplicative semi-metric space. *Journal of Scientific Research*, 2020, *12*(3), 341–348.

❐❐❐