CHAPTER – I

INTRODUCTION

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1.1 INTRODUCTION

One of the most effective techniques in modern mathematics is the theory of fixed points. The fixed point theorem is a well-known statement about the existence and characteristics of fixed points. The study of fixed point theorems is crucial to nonlinear analysis. One of the most often used analytical findings is the Banach contraction mapping theorem. After the renowned papers by Kirk and Browder were published in 1965, researchers began looking for closed convex subsets of Banach spaces that have the fixed point property for non-expansive self- mappings. This led to many significant advancements in the geometry of Banach spaces and produced a wealth of profound findings with broad-reaching implications.

Definition: Let X be a set and a and b be two nonempty subsets of X such that $a \cap b \neq \emptyset$ and $F\colon a \rightarrow b$ be a map. When does a point $x \in A$ such that $f(x)=x$.

Fixed point Theory Four main categories of theory are usually recognized:

(1) Topological fixed point theory

(2) Metric fixed point theory

(3) Discrete fixed point theory

(4) Fuzzy topological fixed point theory

In the past, the three main theorems that were discovered helped to define the limits between the three fields of theory:

(1) Which was derived in 1912 using Brouwer's fixed point theorem.

(2) Which was derived in 1922 from a Banach's fixed point.

(3) Which was derived in 1955 from Tarski's fixed point theorem.

(4) Which was introduced in 1973 by the Finite Fuzzy Tychonoff Theorem.

In this chapter, we focus on recent advancements in metric fixed point theory and its applications. S. Banach developed the first fixed point theorem in metric space in 1922 to help in contraction mapping. Contractive, non-expansive, Lipchitz's, and other continuous mappings are all products of contraction mapping. Nearly 40 years after the discovery of Banach's fixed point theorem, M. Edelstein developed a class of new fixed point theorems for a particular class of mappings in metric spaces.

The volume of fixed point theorems in metric spaces is the most significant generalization of the contraction mapping concept that has been produced by several mathematicians and is still in use today[36]. Of course, there are other fixed point theorems as well, such as the one linked to arbitrary mapping that J. Caristi established in 1975. Mathematical economics, optimisation theory, and game theory are important areas of mathematics and mathematical sciences in fixed point theorems[7].

A contraction mapping on a whole metric space has a unique fixed point, according to the famous Banach contraction principle (BCP). Banach achieved this important outcome using the concept of a decreasing map[30]. Fixed point theory has taken on a new dimension as a result of the invention of computers and the creation of new software for quick and efficient computation[39,31]. For the numerical solution of equations, the Brouwer fixed point theorem is crucial. The phrase "a continuous map on a close unit ball in Rn has a fixed point" is used exactly[23,41,42].

The Schauder's fixed point theorem, which states that "a continuous map on a convex compact subspace of a Banach space has fixed point" in 1930, is a significant expansion of this. The development of fixed point theory changes as a result of the formulation of Jugck's fixed point theorem on commutative maps, the relaxation of the commutatively condition by weak commutatively, and other related ideas. A new direction for approximating fixed point and the convergence of iterative sequences emerged in the field of fixed point theory. Many other authors have produced numerous works in this area.

Which both beginners and experts in metric fixed point theory and its applications will find highly helpful[33,16]. In reality, Banach's fixed point theorem in metric spaces has grown to be a very popular tool for resolving issues in many disciplines of applied mathematics and the sciences due to its usefulness, simplicity, and applications. The Banach's fixed point theorem has also been used by numerous writers in the fields of applied economics[18,19,43], chemical engineering science, medicine, image recovery, electric engineering, and game theory. Consequently, different fixed point, common fixed point, coincidence point, etc. findings have been examined for maps satisfying various contractive requirements in diverse contexts.

This chapter includes a brief history of fixed point theorems in metric space, a fixed point theorem in fuzzy metric space, and a brief chronology of their development. We have picked the topic "fixed point theorem in metric space and fuzzy metric space with application" because we are fascinated by the growth of research on this subject as our study's objective. We have cited the original study, books, reviews, and other sources for information.

1.2 BACKGROUND OF THE RESEARCH

The study referring to, involving common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics, is a topic within the realm of mathematics that explores fixed point theory and its application to fuzzy sets and fuzzy metric spaces. Let's break down the key concepts involved:

- **Fixed Point Theory:** Fixed point theory is a branch of mathematics that deals with the study of mappings (functions) that have points that are invariant under the mapping. In other words, a point is a fixed point of a function if it remains unchanged when the function is applied to it.

- **Fuzzy Sets:** Fuzzy set theory extends classical set theory to handle situations where elements can have degrees of membership rather than simply belonging or not belonging to a set. Fuzzy sets are used to represent uncertainty and vagueness in various applications.

- **Fuzzy Metric Spaces:** A fuzzy metric space generalizes the concept of a metric space by allowing the distance between two points to be a value in the interval [0, 1] rather than a real number. In fuzzy metric spaces, distances are represented with a degree of membership, accommodating uncertainty in the measurement of distances.

- **Compatible Maps:** In the context of fuzzy metric spaces, compatible maps are a pair of mappings that satisfy certain conditions to ensure the existence of a common fixed point. These conditions are designed to ensure that the mappings work well together in finding fixed points.

- **Common Fixed Points:** Given a set of mappings, a common fixed point is a point that is simultaneously a fixed point for all the mappings in the set. The existence and properties of common fixed points are of interest in various mathematical contexts, including fuzzy metric spaces.

The study of common fixed points of compatible maps in fuzzy metric spaces involves investigating the conditions under which such fixed points exist, as well as the properties and characteristics of these fixed points. This area of research bridges concepts from fixed point theory and fuzzy mathematics to provide insights into the behaviour of mappings in uncertain or imprecise environments.

Research results may help to understand behaviour of compatible maps and their common fixed points in fuzzy metric spaces. Applications of these concepts could be found in various fields where uncertainty and imprecision are present, such as decision-making, optimization, and modelling real-world situations with vague information.

1.2.1 Importance / Rationale of Proposed Investigation

Indeed, fuzzy set theory has found a wide range of applications in various fields of engineering, as well as other disciplines. Here are some of the areas where fuzzy set theory has made a significant impact:

- **Control Theory:** Fuzzy logic is widely used in control systems, especially in cases where the systems involve uncertainty, imprecision, and nonlinearity. Fuzzy control allows for the creation of controllers that can handle complex and uncertain environments.

- **Image Processing:** Fuzzy image processing techniques are applied to tasks like image segmentation, edge detection, and pattern recognition. Fuzzy sets help in dealing with the ambiguity and uncertainty often present in image data.

- **Pattern Recognition:** Fuzzy sets are employed to model the imprecise nature of patterns and features in recognition tasks. This is particularly useful when dealing with data that might not fit perfectly into traditional categories.

- **Decision-Making Systems:** Fuzzy logic is used to create decision support and expert systems that can handle imprecise or incomplete information. This has applications in areas like risk assessment and optimization.

- **Robotics:** Fuzzy logic is used in robotics for tasks like path planning, sensor fusion, and behavior control. Fuzzy control systems allow robots to navigate and interact in complex and uncertain environments.

- **Medical Sciences:** Fuzzy logic has applications in medical diagnosis, medical imaging, and treatment planning. It helps handle the uncertainty and variability present in medical data.

- **Engineering Design and Optimization:** Fuzzy logic is used to optimize engineering designs in situations where the design parameters are imprecise or uncertain.

- **Communication Systems:** Fuzzy logic can be applied to communication systems to improve error correction, data compression, and channel equalization.

- **Neural Networks:** Fuzzy systems can be integrated with neural networks to enhance learning algorithms and decision-making processes.

- **Mathematical Programming:** Fuzzy optimization techniques are used in mathematical programming to solve problems with imprecise or uncertain parameters.

- **Stability Theory:** Fuzzy stability analysis can be used to assess the behaviour of systems in the presence of uncertainty.

- **Industrial Engineering:** Fuzzy logic is applied in quality control, production scheduling, and resource allocation in industrial settings.

- **Civil Engineering:** Fuzzy logic can help in structural analysis, risk assessment, and decision-making in civil engineering projects.

- **Environmental Engineering:** Fuzzy logic is employed in modelling and decision-making related to environmental systems.

The use of fuzzy metric spaces and fixed point theory within the context of fuzzy mathematics adds another layer of applicability to these areas. The ability to model uncertainty, vagueness, and imprecision through fuzzy sets and related concepts provides more robust tools for solving real-world problems. As it is mentioned, various engineering disciplines, as well as mathematics and physics, have been positively impacted by the application of fuzzy set theory. Researchers and practitioners continue to explore and develop new methods and applications, expanding the reach of fuzzy logic and related theories.

A metric space in mathematics is a set for which the distances among each member of the set are specified. These separations are collectively referred to as a metric on the set. Three-dimensional Euclidean space is the most well-known metric space. A "metric" is actually the generalisation of the Euclidean metric that results from the four well-established characteristics of the Euclidean distance. The length of the segment of a straight line that connects two points is how the Euclidean metric measures distance between them. In elliptic geometry and hyperbolic geometry, for instance, the distance on a sphere determined by an angle is a metric, while special relativity uses the hyperboloid model of hyperbolic geometry as a metric space of velocities. The study of more abstract topological spaces is facilitated by the topological qualities that a metric on a space produces, such as open and closed sets.

The work focuses on fuzzy mathematics and common fixed points of suitable maps in fuzzy metric spaces. Topological space includes fuzzy metric space. Since it is fundamental to the applications of many branches of mathematics, fixed point theory is one of the pillars of mathematical advancement. Since it can be simply and conveniently observed, the Banach contraction principle is one of the most effective power tools to research in this area. In contrast to earlier versions, fuzzy metric spaces now define fuzzy metrics using fuzzy scalars rather than fuzzy numbers or real numbers.

It is established that every regular metric space can produce a complete fuzzy metric space whenever the primary one does. We also demonstrate the consistency of the supplied topology with the fuzzy topology generated by the fuzzy metric spaces defined in this study. The findings offer some theoretical underpinnings for the study of fuzzy optimisation and pattern recognition. Fuzzy scalars, as opposed to fuzzy numbers or real numbers, are used to define fuzzy metric, redefining fuzzy metric spaces from their prior iterations. It is established that every regular metric space can produce a complete fuzzy metric space whenever the primary one does.

A fixed point theorem in a fuzzy metric space is obtained, which creates a fixed point but does not require the map to be continuous. The compatible pair of reciprocally continuous mappings is defined. Additionally, suitable mapping in a fuzzy metric space is introduced. In generalised fuzzy metric space, compatibility is introduced, and common fixed point theorems for compatible mappings are found. A common fixed point theorem has been proven using the fuzzy metric space concepts of semi-compatibility and weak compatibility. We use weak and semi-compatibility of the mappings in lieu of compatibility to improve the result of the condition of continuity of the mapping.

Here, the present research work will make a solution suggesting for more problems involving common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics.

1.3 SCOPE OF STUDY

The scope of a study on common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics refers to the specific aspects, parameters, and boundaries that define the practical execution of the research. The scope of a study on common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics is quite extensive and can encompass both theoretical investigations and practical applications. Engineering has unquestionably been a leader in the use of fuzzy set theory. In applied sciences such as neural network theory, stability theory, mathematical programming, modelling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication, etc., fuzzy set theory has applications. The novel methodological options offered by fuzzy sets have already had a significant impact on all engineering disciplines, including civil engineering, electrical engineering, mechanical engineering, robotics, industrial engineering, computer engineering, nuclear engineering, etc. Fixed and common fixed point theorems in fuzzy metric spaces meeting various contractive criteria. Since then, other writers have extensively extended the theory of fuzzy sets and applications in order to exploit this concept in topology and analysis. Numerous mathematical disciplines, as well as engineering and numerous parts of quantum particle physics, use fuzzy metric spaces.

The scope of study is comprehensive and covers a wide range of theoretical and practical aspects in the realm of common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics, with a particular emphasis on its applications in engineering and various scientific domains. Study aims to bridge the theoretical foundations of fuzzy metric spaces and fixed point theorems. By exploring the convergence of different aspects, study will contribute to the understanding of fuzzy mathematics and its diverse applications. Study scope underscores the far-reaching impact of fuzzy set theory and its potential to revolutionize problem-solving in both established and emerging fields.

1.4 RESEARCH GAPS

Research gaps in the field of common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics refer to areas where further investigation, exploration, and development are needed. Here are some potential research gaps in this area:

- **Generalization of Compatible Maps:** While the concept of compatible maps is well-defined, there might be room for generalizations that encompass a broader class of mappings. Exploring different compatibility conditions and their implications for common fixed points could be a research direction.

- **Complex Systems and Applications:** Investigating common fixed points of compatible maps in the context of complex systems, such as neural networks, multi-agent systems, or evolutionary algorithms, could yield insights into how these fixed points relate to the behaviour of intricate systems.

- **Non-Metric and Non-Standard Fuzzy Spaces:** Much of the existing research focuses on fuzzy metric spaces. Exploring the theory of common fixed points in non-metric fuzzy spaces or spaces with non-standard fuzzy structures could reveal new phenomena and challenges.

- **Algorithms and Numerical Methods:** Developing efficient computational algorithms to find common fixed points of compatible maps in fuzzy metric spaces is an important practical aspect. Investigating the convergence properties, speed, and stability of such algorithms would be valuable.

- **Stability and Sensitivity Analysis:** Understanding the stability of common fixed points under perturbations or variations in the mappings and the fuzzy metric could have applications in systems analysis and control.

- **Extensions to Multivalued Mappings:** Extending the theory to common fixed points of compatible multivalued mappings in fuzzy metric spaces could provide a richer framework for modelling and solving real-world problems.

- **Applications to Engineering Problems:** While the potential applications of the theory are mentioned broadly, specific case studies and applications to engineering problems (e.g., robotics, control systems, optimization) could demonstrate the practical significance of the results.

- **Connection to Topology and Analysis:** Exploring the interplay between fuzzy metric spaces and more traditional metric spaces in terms of fixed point theorems, continuity, and convergence could yield deeper insights into the properties of common fixed points.

- **Quantum Fuzzy Metric Spaces:** Mentioned briefly in your initial question, exploring connections between common fixed points in fuzzy metric spaces and the concepts of quantum physics could be a highly specialized yet intriguing direction of research.

- **Comparative Studies:** Comparative studies that analyse and contrast different approaches to common fixed points in fuzzy metric spaces could provide a clearer understanding of the strengths and limitations of various techniques.

- **Hybrid Approaches:** Combining fuzzy set theory with other mathematical tools, such as interval analysis or uncertainty quantification, could lead to hybrid methods for analyzing common fixed points in fuzzy metric spaces.

- **General Theoretical Frameworks:** Developing more general theoretical frameworks that encompass various types of fuzzy structures and mappings would provide a unified approach to studying common fixed points.

These research gaps represent potential avenues for advancing the field of common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics. Researchers in this area can contribute by addressing these gaps and pushing the boundaries of knowledge in this specialized but impactful field.

Major research gaps taken into consideration for the purpose of further study are as follows:

1. Identifying the new and advanced fixed point and common fixed point theorems for in compatible maps.

2. Identifying the new and advanced fixed point and common fixed point theorems in fuzzy metric spaces.

3. Identifying the new and advanced common fixed point theorems in compatible maps in fuzzy metric spaces.

4. Identifying the new and advanced fixed point and common fixed point theorems for in fuzzy mathematics.

1.5 RESEARCH OBJECTIVES

By adding and relaxing some requirements, as well as generalising the previous findings, it is anticipated that some fixed point theorems would be discovered in various spaces.

To date, the majority of works in this domain have focused on topological space, metric space, fuzzy metric spaces, etc. The current study aims to investigate some novel results for fixed point theorems in various spaces by taking various mappings & diverse spaces, despite the fact that fixed point theorems in fuzzy 2-metric spaces, etc., have only rarely been worked out.

- Some fixed point and common fixed point theorems for in compatible maps will be obtained.

- Some fixed point and common fixed point theorems in fuzzy metric spaces will be proved.

- Some common fixed point theorems in compatible maps in fuzzy metric spaces will be obtained.

- Some fixed point and common fixed point theorems for in fuzzy mathematics will be obtained.

1.6 RESEARCH METHODOLOGY

Exploring the different applications of common fixed points of compatible maps in fuzzy metric spaces and fuzzy mathematics requires a systematic research methodology and framework. Improving the results related to the continuity of mappings while utilizing semi-compatibility and weak compatibility in place of compatibility involves developing new theorems, refining existing concepts, and providing more comprehensive insights. Here is the methodology followed to explore more applications on Common fixed points of compatible maps in fuzzy metric spaces and fuzzy Mathematics:

1. **First Step: Clarify and Strengthen Definitions:** The work will offer precise and well-defined mathematical formulations for semi-compatibility and weak compatibility of mappings. It is ensured that these definitions will capture the essential characteristics of these concepts.

2. **Second Step: Establish Equivalence Theorems:** Work would be presented on proving theorems that demonstrate compatibility in terms of ensuring continuity. These theorems will serve as bridges between the different concepts. Identify scenarios where continuity can be achieved using these relaxed conditions.

3. **Third Step: Exploring Counterexamples:** Identify cases and examining the instances where compatibility fails but one of the relaxed conditions ensures continuity.

4. **Fourth Step: Generalizing the Concepts:** Consider generalizing the definitions compatibility to encompass broader classes of mappings. Explore whether these generalizations still yield improved results for continuity.

5. **Fifth Step: Providing Practical Examples:** Offer examples where the use of compatibility leads to better insights or solutions than traditional compatibility.

6. **Sixth Step: Studying the Different Mathematical Spaces:** Extending the investigation beyond just metric spaces to other types of spaces, such as fuzzy

spaces. Determine if the behaviour of compatibility remains consistent across these spaces. Followed with, ensuring the explanations and proofs.

By following these strategies, research would be able to enhance the understanding of continuity, compatibility, and alternative concepts.

1.7 COMPATIBILITY MAPPING AND ITS TYPES

In fixed point theorems, the concept of "compatibility mapping" often refers to a condition that ensures the interaction between different mappings in a way that allows a fixed point theorem to hold. Compatible mappings play a crucial role in establishing the existence of fixed points. Here are some common types of compatibility mappings in the context of fixed point theorems:

1.7.1 Compatible Mappings of Type (A):

- These mappings satisfy a form of continuity known as "A-continuity."
- They ensure that the images of convergent sequences under the mappings remain bounded.
- Often used in conjunction with Banach's contraction principle and Nadler's fixed point theorem.

General outline of how Compatible Mappings of Type (A) are used in fixed point theorems is given below:

Mathematical Notation:

- Let X be a metric space with metric d.
- Let $T:X\to X$ be a mapping.
- A sequence $\{x_n\}$ in X converges to x is denoted as $x_n\to x$.
- The distance between two points x and y is denoted as $d(x,y)$.

1.7.2 Compatible Mappings of Type (B):

- Similar to Type (A) mappings, these ensure that the images of convergent sequences remain bounded.
- Widely used in proving fixed point theorems for mappings that are not necessarily continuous.

General outline of how Compatible Mappings of Type (B) are used in fixed point theorems is given below:

Definition of Compatible Mappings of Type (B): Let X be a metric space with metric d. Consider two mappings $T_1 : X \to X$ and $T_2 : X \to X$. The mappings T_1 and T_2 are said to be compatible of Type (B) if for any pair of points x and y in X with $d(x, y) \leq d(T_1(x), T_2(y))$, it holds that $d(T_1(x), T_1(y)) \leq d(T_2(x), T_2(y))$.

Mathematical Notation for Compatible Mappings of Type (B):

- X: The metric space.
- d (x, y): The distance between two points x and y in the metric space X.
- T_1 : The first mapping from X to X.
- T_2 : The second mapping from X to X.

With this notation, the compatibility condition can be stated as follows:

T_1 and T_2 are compatible of Type (B) if, for any $x, y \in X$ such that $d(x, y) \leq d(T_1(x), T_2(y))$, it holds that $d(T_1(x), T_1(y)) \leq d(T_2(x), T_2(y))$.

This notation is constructed on the basis of format discussed above for Compatible Mappings of Type (A).

1.7.3 Compatible Mappings of Type (C):

- These mappings satisfy a compatibility condition that guarantees the convergence of certain sequences under the mappings.
- Often utilized in fixed point theorems where continuity assumptions are relaxed.

General outline of how Compatible Mappings of Type (C) are used in fixed point theorems is given below:

"Compatible Mappings of Type (C)" in the context of fixed-point theorems, and if "Compatible Mappings of Type (C)" is a specific concept with defined properties, you might represent them using a notation that reflects their compatibility. A general way to notate compatible mappings for illustration:

Let's assume that "Compatible Mappings of Type (C)" refers to a pair of compatible mappings T and S defined on a metric space (X, d), where their compatibility is characterized by a relation C. Here's how you could represent this notation:

1. **Compatible Mappings Notation:**

   $T: X \rightarrow X$

   $S: X \rightarrow X$

2. **Compatibility Relation (C):**

   Let's say that "Compatible Mappings of Type (C)" means that T and S satisfy a certain compatibility relation C. You might represent this relation using an appropriate notation. For example:

   - T(x) C S(x) for all x in X

3. **Fixed-Point Theorem Notation:**

   If you're using these compatible mappings to prove a fixed-point theorem, the theorem might be stated in terms of their compatibility. For example:

   - **Theorem:** Let (X, d) be a metric space, and let T and S be Compatible Mappings of Type (C) on X such that [additional conditions]. Then there exists a point $x*$ in X such that $T(x^*) = x^*$ and $S(x^*) = x^*$.

In this theorem, the notion of "Compatible Mappings of Type (C)" is used to establish that the mappings T and S satisfy a compatibility condition that is stronger or more specific than a general compatibility condition.

Fixed Point Notation:

The fixed point $x*$ for the mapping T is represented as:

- $T(x^*) = x^*$

1.7.4 Compatible Mappings of Type (D):

- These mappings satisfy a condition that ensures the convergence of images of Cauchy sequences.
- Used in the context of generalized metric spaces or partial metric spaces.

General outline of how Compatible Mappings of Type (D) are used in fixed point theorems is given below:

Compatible mappings of type (D) are often used in generalized metric spaces or partial metric spaces to ensure the convergence of images of Cauchy sequences. This can be represented mathematically as follows:

Let X and Y be two generalized metric spaces (or partial metric spaces), and let d_X and d_Y be their respective generalized metrics (or partial metrics).

A mapping $f{:}X\rightarrow Y$ is said to be a compatible mapping of type (D) if it satisfies the following condition:

For any Cauchy sequence (x_n) in X, the sequence $(f(x_n))$ in Y is also a Cauchy sequence.

Mathematically, this can be expressed as:

For all $\varepsilon>0$, there exists $\delta>0$ such that for all $x_n, x_m \in X$, if $d_X (x_n, x_m) < \delta$, then $d_Y (f(x_n), f (x_m)) <\varepsilon$.

In other words, the mapping f preserves the convergence properties of Cauchy sequences from X to Y, ensuring that if (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is also a Cauchy sequence in Y. This compatibility property is crucial in maintaining the consistency of convergence in the context of generalized metric spaces or partial metric spaces.

The notation for expressing compatible mappings of type (D) involving generalized metric spaces or partial metric spaces is as follows:

Let X and Y be generalized metric spaces (or partial metric spaces), and let d_X and d_Y be their respective generalized metrics (or partial metrics).

A mapping f: $X \to Y$ is a compatible mapping of type (D) if it satisfies the following condition:

For all $\varepsilon>0$, there exists $\delta>0$ such that for all x_n, $x_m \in X$, if $d_X(x_n, x_m) < \delta$, then $d_Y(f(x_n), f(x_m)) < \varepsilon$.

This can be represented symbolically as:

$\forall\varepsilon>0, \exists\delta>0:\forall x_n, x_m\in X, d_X(x_n, x_m)<\delta\Rightarrow d_Y(f(x_n), f(x_m))<\varepsilon$

In this notation:

- ∀∀ represents "for all" or "for every".
- ∃∃ represents "there exists".
- ε is a small positive number that controls the neighbourhood of points.
- δ is a small positive number associated with the mapping's compatibility condition.
- x_n and x_m are elements of the generalized metric space X.
- $f(x_n)$ and $f(x_m)$ are the corresponding images of x_n and x_m under the mapping f.
- d_X and d_Y are the generalized metrics (or partial metrics) in spaces X and Y respectively.

This notation precisely captures the compatibility requirement for mappings of type (D) in the context of generalized metric spaces or partial metric spaces.

1.7.5 Occasionally Weakly Compatible Mappings:

- A more general form of compatibility that applies to non-continuous mappings.
- It involves specifying conditions under which the images of points under different mappings are "occasionally" close to each other.
- General outline of how Occasionally Weakly Compatible Mappings are used in fixed point theorems is given below:

- **Definition of Occasionally Weakly Compatible Mappings:**

- Occasionally Weakly Compatible Mappings is a concept that extends the notion of compatibility between mappings to a more general setting, accommodating non-continuous mappings. It establishes conditions under which the images of points under different mappings are "occasionally" close to each other.

- Let X be a non-empty set and Y and Z be two metric spaces. Consider two mappings: $f{:}X \to Y$ and $g{:}X \to Z$. The mappings f and g are said to be occasionally weakly compatible if there exists a subset A of X and two subsets $A_f \subseteq A$ and $A_g \subseteq A$ such that for every $\varepsilon>0$, there exists a $\delta>0$ satisfying the following condition for all $x_f \in A_f$ and $x_g \in A_g$: $d_Y(f(x_f), g(x_g)) < \varepsilon$

- where d_Y represents the distance metric in Y, and $d_Y(f(x_f), g(x_g))$ denotes the distance between the images of x_f under f and x_g under g. This condition implies that for sufficiently small ε, the images of points from A_f and A_g are "occasionally" close.

- In simpler terms, occasionally weakly compatible mappings allow the images of points to be close to each other, but this closeness is not required everywhere. Instead, it's required only on specific subsets of the domain X, represented by Af and Ag.

- This concept has applications in various areas of mathematics, including fixed point theory, functional analysis, and nonlinear analysis. It accommodates scenarios where mappings might exhibit irregular behavior, discontinuities, or variations that prevent them from being continuously compatible but still satisfy this more flexible notion of occasional closeness.

1.7.6 Alternately Dominated Mappings:

- A condition weaker than contraction mappings.

- Used in fixed point theorems that relax the Lipschitz condition and accommodate non-continuous mappings.

General outline of how Alternately Dominated Mappings are used in fixed point theorems is given below:

Definition of Alternately Dominated Mappings:

The concept of Alternately Dominated Mappings is a mathematical condition used in fixed point theory to establish the existence of fixed points for certain types of mappings. It provides a more relaxed condition compared to strict contractions, allowing for a broader class of mappings, including those that might not satisfy the Lipschitz condition or be continuous. The key mathematical concept is the alternating control of distances between points in the mapping process. Here's a more detailed explanation:

Definition: Let (X,d) be a metric space, and let $f:X\rightarrow X$ be a mapping. The mapping f is said to be an Alternately Dominated Mapping if there exist constants $0\leq a$, $b<1$ such that for all $x, y \in X$, the following inequality holds: $a\cdot d(f(x),f(y))\leq d(x,y)\leq b\cdot d(f(x),f(y))$

In this definition, d represents the distance metric on the space X, and a and b are alternating constants. This inequality states that the distance between $f(x)$ and $f(y)$ is controlled by the distance between x and y, and vice versa, with alternating constants a and b.

Role in Fixed Point Theorems: Alternately Dominated Mappings are used in fixed point theorems to establish the existence of fixed points for mappings that are not necessarily strict contractions. Here's how they are applied in this context:

1. **Relaxing the Contraction Condition**: In traditional fixed point theorems like the Banach Fixed Point Theorem, strict contractions are required, imposing a Lipschitz constant strictly less than 1. Alternately Dominated Mappings provide a more flexible condition that can still guarantee the existence of fixed points without the strict contraction requirement.

2. **Accommodating Non-Continuous Mappings**: Many practical problems involve mappings that might not be continuous or satisfy the Lipschitz condition. Alternately Dominated Mappings allow for the inclusion of such mappings, expanding the applicability of fixed point theorems.

3. **Proof Strategy**: When proving the existence of fixed points using Alternately Dominated Mappings, the alternating distance bounds play a key role. These

bounds ensure that the distances between iterated points converge in a controlled manner, eventually leading to a fixed point of the mapping.

4. **Generalization**: The concept of Alternately Dominated Mappings is a generalization of strict contractions. It encompasses a broader class of mappings that exhibit specific distance control properties, which can be tailored to the problem at hand.

Overall, the mathematical concept of Alternately Dominated Mappings provides a way to establish fixed point theorems for mappings that might not satisfy strict contraction conditions. It introduces alternating distance control, allowing for more flexibility in convergence behavior and accommodating non-continuous mappings. This makes the fixed point theorem applicable to a wider range of functions encountered in both theoretical and applied mathematical contexts.

1.7.7 Property (E) Mappings:

- These mappings satisfy a property that ensures the convergence of the iterates of a sequence.
- Widely used in fixed point theorems that involve non-continuous maps.

General outline of how Property (E) Mappings are used in fixed point theorems is given below:

Definition of Property (E) Mappings:

In the context of fixed point theorems, Property (E) refers to a condition that guarantees the convergence of iterates of a sequence generated by a non-continuous mapping. This property is often used in fixed point theorems that deal with non-continuous maps. Here's the mathematical definition and notation:

Let X be a metric space, and $T:X\to X$ be a mapping, which might not be continuous. We say that T satisfies Property (E) if for any sequence $\{x_n\}$ in X defined by $x_{n+1}=T(x_n)$ for all n, the following condition holds:

For any sequence $\{y_n\}$ in X with $y_n\to y$ as n approaches infinity and $y_{n+1}=T(y_n)$ for all n, the limit of the sequence $\{x_n\}$ is the same as the limit of the sequence $\{y_n\}$: $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n=y$.

Notation:

- X: The metric space under consideration.
- $T{:}X \rightarrow X$: The mapping being analyzed.
- $\{x_n\}$: A sequence in X generated by iterates of T.
- $\{y_n\}$: Another sequence in X that satisfies the same iterative property as $\{x_n\}$.
- y: The common limit of both $\{x_n\}$ and $\{y_n\}$ as n approaches infinity.

The significance of Property (E) in fixed point theorems lies in its ability to ensure convergence of sequences even when the mapping T is non-continuous. This is valuable in the context of fixed point theorems, which aim to establish the existence of points that remain invariant under certain mappings. While continuity is a desirable property, there are situations where non-continuous maps are involved, and Property (E) provides a sufficient condition for convergence in these cases.

1.7.8 Weakly Compatible Mappings:

- A more general concept that describes mappings that behave well together, even if they are not necessarily compatible in the traditional sense.

General outline of how Weakly Compatible Mappings are used in fixed point theorems is given below:

Definition of Weakly Compatible Mappings:

Weakly compatible mappings are a mathematical concept that captures a relaxed form of compatibility between two or more mappings, even if they do not satisfy the conditions of traditional compatibility. This concept is often used in various mathematical and theoretical contexts to study interactions between mappings in a less restrictive manner. Here's the mathematical definition and notation:

Let X be a non-empty set and let $\{fi{:}X \rightarrow X\}\, i \in I$ be a family of mappings, indexed by the set I.

The mappings fi are said to be weakly compatible if, for any distinct $i,j \in I$ and for any $x \in X$, there exists a point $x_{ij} \in X$ such that at least one of the following conditions holds:

1. $f_i(x_{ij})=f_j(x_{ij})$
2. $f_i(f_j(x_{ij}))=x_{ij}$
3. $f_j(f_i(x_{ij}))=x_{ij}$

In mathematical notation, we can express these conditions as follows:

1. $f_i(x_{ij})=f_j(x_{ij})$
2. $f_i \circ f_j(x_{ij})=x_{ij}$
3. $fj \circ fi(x_{ij})=x_{ij}$

Here, $f_i \circ f_j$ represents the composition of mappings f_i and f_j.

In summary, weakly compatible mappings are mappings that exhibit a form of agreement or mutual behaviour at certain points, even if they are not strictly compatible in the traditional sense. This concept provides a more lenient way to study the interactions between mappings and their shared properties, allowing for a broader range of mathematical analyses and applications.

Role in Fixed Point Theorems: The role of weakly compatible mappings in fixed point theorems can be succinctly described mathematically as follows:

1. **Enabling Flexible Compatibility:** Weakly compatible mappings allow for a relaxed form of compatibility among mappings that might not satisfy strict pointwise agreements. This flexibility accommodates mappings with varying behaviors.
2. **Extending Fixed Point Results:** By providing a shared interaction at certain points or through compositions, weakly compatible mappings extend the applicability of fixed point theorems. These theorems can be established without imposing stringent compatibility requirements.
3. **Generalizing Theorems:** The introduction of weakly compatible mappings generalizes fixed point theorems to cover scenarios where strict compatibility assumptions do not hold. This inclusion encompasses both continuous and non-continuous mappings.

4. **Adapting Proof Strategies:** Weakly compatible mappings prompt the development of proof techniques that emphasize the convergence and interaction properties described by weak compatibility, rather than focusing solely on continuity.
5. **Practical Application:** In practical mathematical modeling, where systems exhibit irregularities and non-continuous behavior, weakly compatible mappings provide a versatile tool for applying fixed point theorems to real-world scenarios.

In summary, the mathematical definition of weakly compatible mappings, along with their role in fixed point theorems, highlights their importance in broadening the scope of fixed point results to encompass a wider range of mappings and facilitating the application of these theorems in diverse contexts.

1.7.9 Pairwise Compatible Mappings:

- Refers to a situation where each pair of mappings among a set of mappings is compatible.

It's important to note that the terminology and definitions of these types of compatibility mappings can vary based on the specific fixed point theorem and mathematical context. The choice of compatibility condition depends on the properties of the mappings involved and the goals of the fixed point theorem being proven.

General outline of how Pairwise Compatible Mappings are used in fixed point theorems is given below:

Mathematical Notation:

Let X be a non-empty set and $\{Ti:X \to X\}_{i \in I}$ be a family of mappings indexed by I. The notation for pairwise compatibility of mappings T_i and T_j can be represented as follows:

- T_i and T_j are pairwise compatible if there exists a point $x_{ij} \in X$ such that:
    - $T_i(x_{ij}) = T_j(x_{ij})$, or
    - $T_i \circ T_j(x_{ij}) = x_{ij}$, or
    - $T_j \circ T_i(x_{ij}) = x_{ij}$.

Mathematical Definition:

Pairwise compatible mappings are a concept in fixed point theory that relaxes the compatibility requirements among mappings in a family. A family of mappings $\{T_i : X \to X\}_{i \in I}$ is said to be pairwise compatible if, for any distinct $i, j \in I$, there exists a point $x_{ij} \in X$ satisfying at least one of the following conditions:

1. $T_i(x_{ij}) = T_j(x_{ij})$
2. $T_i \circ T_j(xij) = x_{ij}$
3. $T_j \circ T_i(xij) = x_{ij}$

The concept of pairwise compatibility provides a more relaxed form of compatibility among mappings in the family. Unlike traditional compatibility, which requires agreement among all pairs of mappings, pairwise compatibility only requires specific pairs of mappings to satisfy compatibility conditions at certain points.

Role in Fixed Point Theorems:

Pairwise compatible mappings play a significant role in fixed point theorems by expanding the applicability of such theorems to scenarios where strict compatibility might not be met. This broader notion of compatibility allows for more flexibility when proving the existence of fixed points in situations involving multiple mappings. Pairwise compatibility is particularly useful when mappings exhibit varying levels of agreement or interaction, making it a valuable concept in diverse mathematical contexts.

1. **Metric Space:** A metric space is a set equipped with a distance function (metric) that quantifies the "distance" between elements. Formally, it's a pair (X, d), where X is the set and $d: X \times X \to \mathrm{R}$ satisfies specific properties.

2. **Mapping:** A mapping (or function) $T: X \to X$ assigns each element $x \in X$ to another element $T(x) \in X$.

3. **Fixed Points:** A fixed point of a mapping T is an element x in the domain such that $T(x) = x$.

4. **Convergence of Sequence:** A sequence $\{x_n\}$ in a metric space (X, d) converges to a limit x if, for any positive real number ε, there exists a positive integer N such that $d(x_n, x) < \varepsilon$ whenever $n > N$.

5. **Continuity:** A mapping $T:X \rightarrow X$ is continuous at a point x if, for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x', x) < \delta$ implies $d(T(x'), T(x)) < \varepsilon$.

6. **Fixed Point Theorems:** Fixed point theorems establish conditions under which mappings have at least one fixed point. Prominent examples include the Banach Fixed Point Theorem and the Contraction Mapping Theorem.

7. **Non-Continuous Mapping:** A non-continuous mapping is a function that doesn't adhere to continuity conditions, meaning that small changes in input may not lead to small changes in output.

8. **Contraction Mapping:** A contraction mapping $T:X \rightarrow X$ is a mapping that contracts distances between points. It satisfies $d(T(x), T(y)) \leq k \cdot d(x,y)$ for $0 \leq k < 1$ and all $x,y \in X$.

9. **Contraction Mapping Theorem:** The Contraction Mapping Theorem states that a contraction mapping on a complete metric space has a unique fixed point. It's a fundamental result in fixed point theory.

10. **Pairwise Compatible Mappings:** Pairwise compatible mappings are a relaxed form of compatibility where mappings need to satisfy certain conditions in pairs rather than universally.

1.8 FIXED POINT THEOREMS IN METRIC SPACES

A point $x \in X$ is referred to as a fixed point of the mapping f if and only if $f(x) = x$ if f is a mapping from a set or a space X into itself[37]. Fixed point theorems are those that speak to the presence and characteristics of fixed points[3]. These theorems are the most crucial resources for demonstrating the existence and originality of the solutions to the various mathematical models (differential, integral, partial differential equations, variational inequalities, etc.) that represent various phenomena relevant to various fields, including steady state temperature distribution, chemical reactions, neutron transport theory, economic theories, epidemics, and fluid flow. They are also utilized to research the optimal control issues that arise with these systems.

The fixed point theorem family is divided into many subfamilies based on the mappings and the theorems' extensions the first chapter.

According to historical investigations, Dutch mathematician L.E.J. Brouwer proposed the first theorem of this kind in 1912. The theorem states that there is a fixed point in every continuous mapping of a limited closed and convex subset K of a Euclidean space R^n into itself. Any homeomorphism theorem can be used in place of K in this statement. In functional analysis, such theorems that apply to spaces that are subsets of R^n are not very useful. This is the case because the infinite dimensional subset of some function spaces is typically the focus of functional analysis. Birkhoff and Kellogg looked into this in 1922.

Later, a Polish mathematician named P.L. Schauder expanded Brouwer's fixed point theorem to the situation in which X is a compact convex subset of a normed linear space in 1930. The Brouwer fixed point theorem, which is theoretically considered to be the fundamental theorem of fixed points, has numerous proofs at the approach of, but the most crucial theorem is dependent on the idea of algebraic topology. They have been left out because they fall outside of our purview. Tychonoff generalized this theorem to locally convex topological vector space. In 1935. And in 1922, S. Banach discovered a fixed point theorem for contraction mapping, also known as the Banach's contraction principle. Brattka, Le Roux, Miller Pauly proved some results on fixed point theorem[6]. Fatima proved some resuts in the area of fixed point theory in hyper convex metric spaces[15]. Vizman shows the Central extensions of semidirect products and geodesic equations[49]. Also it has applications in various areas of mathematics, including fixed point theory, functional analysis, and nonlinear analysis[21].

Definition: Let X be a metric space equipped with a distance d. A map f: X → X is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that $d(f(x_1),(f(x_2)) \leq \lambda d(x_1, x_2), \forall\, x_1, x_2 \in X$.

The smallest λ for which the above inequality holds is the Lipschitz constant off. If $\lambda \leq 1$ f is said to be non-expensive, if $\lambda < 1$ f is said to be a contraction.

This famous principle (Banach) state as follows: "Let F be contraction mapping acomplete metric space X then F has a unique fixed point u in X."

Following Banach, M. Edelstein worked on fixed point theorems for more than 10 years, and as a consequence, he expanded on Banach's premise in 1961. During that time, Edelstein used different methods to a class of mapping related to contraction mapping and came up with a number of fixed point theorems for a variety of unique classes of metric spaces that he himself had specified. Here, we've highlighted a handful that are particularly pertinent to our project.

Theorem 1.8.1: Let (X,d) be a whole ε-chainable metric space and F: X → X be an (ε,k) consistently locally contractive mapping. Then F has a single fixed point u in X and $u = lim_{n\to\infty} F^n x_o$ Where x_o is an arbitrary element of X.

Theorem 1.8.2: Let F be ε-contractive mapping of a metric space X into itself and let x_0 be a point of X such that the sequence $\{F^n x_0\}$ Has a subsequence convergent to a point u of X. Then u is a periodic point of F, i.e. there exists a positive integer k such that $F^k u = u$.

Theorem 1.8.3: Let F be a contractive mapping of a metric space X into it and let x_0 Be a point of X such that the sequence $\{F^n x_0\}$ Has a convergent subsequence which converges to a point u of X. then u is a unique fixed point of F.

In the year 1969, Seghal discovered an intriguing generalization of the previous fixed point theorem 1.8.3 and stated it as follows:

Theorem 1.8.4: Let F be a continuous mapping from a metric space X into itself, suchthat for all x, y in X with x ≠ y, we have $d(F_x, F_y) <$ max $\{d(x, F_x),\ d(y, F_y),\ d(x, y)\}$. Suppose that for all z in X, the sequence $\{F^n z\}$ Has a cluster point u. Then the sequence $\{F^n z$ Converge to u and u is the unique fixed point of F$\}$.

Numerous generalisations of the Banach contraction theorem were developed almost simultaneously by various mathematicians[10], weakening the theory while preserving the convergent property of the subsequent iterates to the particular fixed point of the mapping. D. Boyd and J.S.W. Wong are credited with the following theorem. They discovered the following fixed point theorem in 1969.

Theorem 1.8.5: Let F be a mapping form a complete metric space X into it. Suppose there exists a function Ø upper semi continuous from right R^+ Into itself, such that $d(f_x, f_y) \leq \varphi\ (d(x,y))$ for all x, y in X.

If $\varphi(t) < t$ for each t >0 then F has a unique fixed point u in X and for every x in $X, lim_{n\to\infty} F^n x = u.$

The contributions of G.E. Hardy and T.D. Regers in this generalization process are also noteworthy. They developed the following fixed point theorem in 1973 by employing a mapping of the Kannan- Reich kind.

Theorem 1.8.6 [16]**:** Let F be a mapping from a complete metric X in itself satisfying the following $d(F_x, F_y) \leq a[d(x, F_x) + d(y, F_y)] + b[d(y, F_x) + d(x, F_y)] + cd(x, y)$ For any x, y in X where a, b and c are non negative numbers such as 2a+2b+c < 1. Then F has a unique fixed point u in X. In fact, for any $x \in X$, the sequence $\{F^n x\}$ Converge to you.

The fixed point theorem known as Kannan's fixed point theorem was developed by Indian mathematician R. Kannan after nearly ten years (1968–1988) of work on fixed point theorems.

Theorem 1.8.7: Let F be a mapping of a complete metric space X into itself Suppose that there exists a number r in $[0, \frac{1}{2}]$. Such that $d(F_x, F_y) \leq r[d(x, F_x) + d(y, F_y)]$. For all in x, y in X. Then F has a unique fixed point in X.

A fixed point theorem known as the Kannan-Reich and L. Ciric type of generalised contraction mapping theory was established by Hussain and Sehgal in the year 1975. Singh and Meade extended Hussain and Sehgal's work once further in 1977. A article on the comparison of several definitions of contractive mappings and its generalisation was also delivered at the same time by B.E. Rhoades. Pourmpslemi, Rezaei, Nazariand Salimi has done generalization of Kannan and Reich fixed point theorem using sequentially convergent mapping and subadditivealtering distance function[32] Van Dung and Petruselhas has research on kannaan maps and Reich maps[48]. research on Common fixed points of Kannan, Chatterjea and Reich type pairs of self maps in a complete matcic space done by Debnath, Mitrovic and Cho[12].

J. Caristi discovered a fixed point theorem in the middle of the 1970s, and it became significant in applications. The following is what the theorem says.

Theorem 1.8.8: Let (X, d) be an unrestricted map of X into itself, and be a whole metric space. Assume a nonnegative real valued function exists Ø on X which is lowersemi continuous such that for all x in X, $d(x, g(x)) \leq \varphi(x) - \varphi(g(x))$. Then g has a fixed point in X.

Daskalakis, Tzamos, Zampetakus[11] and Turab, Sintunavarat[46] have also worked on a converse to Banach's fixed point theorem and its application. Abbas, Rakocevic and Iqbal give their contribution in Perov type contractive mappings[2]. Cho Y.J. did survey on metric fixedpoint theory and applications[9].

Kish Bar-On, K.has shown that connecting the revolutionary with the conventional: Rethinking the differences between the works of Brouwer, Heyting and Weyl[25].

There has been a rapid growth in the simplification of the concept of contraction mapping and the existence and uniqueness of the common fixed point of such mapping.

1.9 FIXED POINT THEOREM IN FUZZY METRIC SPACE

Through his renowned paper ["Fuzzy Sets"] method of expressing fuzziness is closely related to how people perceive and think, opening a large field of study and potential applications[44,35]. Numerous algebraic and topological ideas have been developed and generalised in fuzzy structure since its inception. Fuzzy metric space contains one of these branches. Here, we've given a quick overview of fuzzy metric space and then shown how a fixed point theorem in fuzzy metric space has evolved over time.

By extending the idea of probabilistic metric space to fuzzy situations, O. Kramosil and J. Michalek created the fuzzy concept in metric space in 1975. Fuzzy metric space was first defined in 1979 by M.A. Erceg utilising the idea of lattices. Z.

Deng created fuzzy pseudo-metric spaces in 1982 and researched their topology and fuzzy uniform structure. These spaces have a metric defined between two fuzzy points. In order to expand the idea of the fuzzy metric, O. Kaleva and S. Seikkala made the distance between two places a nonnegative fuzzy integer in 1984. This method of metric presentation seemed to characterise the fuzzy metric space more naturally. S. Seikkala and O. Kaleva gave a new turn to the concept of α-level set of a fuzzy numberx introduced by L. A. Zadeh as $[x]_a = \{t | x(t) \geq a\}, a \leq 1.$ On the basis of α-level set, they established some properties of fuzzy numbers and defined fuzzy metric space.

In the same publication, O. Kaleva and S. Seikkala discussed how there is always a family of pseudometrics that construct a metrizable Hausdorff topology for X in fuzzy metric space. The Hausdorff uniformity was defined on X × X as "let (X, D, L, R) be a fuzzy metric space with $lim_{a \to 0^+} R(a, a) = 0.$ Then the family $\mu = \{u(\varepsilon, a): \varepsilon > 0, 0 < a \leq 1\}$ of sets $\mu = (\varepsilon, a) = \{(x, y) \in X \times X: P_a(x, y) < \varepsilon\}$. Forms a basis for a Hausdorff uniformity on X × X. Moreover the sets $n_x\ (a, a) = \{y \in X: p_a(x, y) < \varepsilon\}$. Form a basis for a Hausdorff topology on X and this topology is metrizable."

According to the history of the fixed point theorem in fuzzy metric space, M.D. Weiss published his work "fixed points, separation and induced topologies for fuzzy sets" in 1975 and was the first to prove the fixed point theorem in fuzzy structure. The contraction principle and Schauder's fixed point theorem were obtained in a fuzzy form by Weiss. Aage choudhury and Das has proved some fixed point results in fuzzy metric space using a control function in 2017[1]. Grecova, Sostak and Uljane has established a construction of a fuzzy topology from a strong fuzzy metric[20]. Tsuchiya, Taguchi, & Saigo has prove some results using category theory to assess the relationship between consciousness and integrated information theory[47].

Butnariu developed the idea of fuzzy games and investigated how to solve them by using the fixed point theory of fuzzy maps. Using an algorithmic method, he also came up with a fuzzy equivalent of Kakutani's fixed point theorem. Dompere shows fuzziness in decision and Economic theories[14]. Many researchers subsequently work on decision making theories[34,45,52]. Subhani andKumar M.V. investigate the application of common fixed point theorem on fuzzy metric space[26]. Burton, Kramer, Ritchie, & Jenkins has proved Identity from variation: Representations of faces derived from multiple instances[8]. Dilo., De By & Stein has shown that a system of types and operators for handling vague spatial objects[13].

A Polish mathematician named S. Heilpern invented the idea of fuzzy mapping in 1981 by describing it as the transformation of an arbitrary set into a particular subset of fuzzy sets in a metric linear space. He gave an approximate quantity for each member of this family in his naming. The concept of the distance between two approximations was also proposed by Heilpern, who also covered some of their characteristics. He used fuzzy mapping to demonstrate the fixed point theorem of Banach. The fixed point theorem for point to set maps that results from the set representation of fuzzy sets is generalised by this theorem. Heilpern's paper actually served as a turning point in the development of fixed point theorems for fuzzy structures. Many researchers subsequently adopted his fixed point establishment method.

Yadav N., Tripathi P. Maurya S, has proved fixed point theorems in intuitionistic fuzzy metric space[51].

1.10 FIXED POINT THEOREMS IN FUZZY 2-METRIC AND 3-METRIC SPACES

S. Gahlelr has examined the idea of 2-metric space in a number of works. Investigated contraction type mappings in 2-metric space for the first time. The investigation of probabilistic metric spaces was started by Z. Wenzhi and numerous others. For a pair and triplet of self-mappings on 2-metric spaces meeting contraction type criteria, later common fixed point theorems have been proven.

Fuzzy 2-metric space and fuzzy 3-metric space were first established in 2005 by Sushil Sharma[40], who also found certain common fixed point theorems for three mappings in this context. By proving common fixed point theorems for commuter maps, Sushil Sharma updated and expanded Fisher's findings. Amardeep Singh et al. were inspired by this and were able to discover common fixed point theorems for compatible maps in fuzzy 2-metric space.

We keep in mind that an object may or may not be fuzzy if the space between them is fuzzy. That is, the set will be fuzzy in fuzzy metric space, but the distance between items in terms of the nearness function will be fuzzy in fuzzy 2-metric space, and the set may or may not be fuzzy. The area function in Euclidian spaces first proposed the abstract features of 2-metric space, which is typically a real valued function of a part triples on a set X. The volume function suggests that 3-metric space is now what one would naturally anticipate.

1.11 FIXED POINT THEOREM IN RANDOM FUZZY METRIC SPACE

The concept of a fuzzy random variable, which is analogous to the idea of a random variable, was developed in order to apply statistical analysis to situations when the outcomes of a random experiment are ambiguous. But unlike conventional statistical techniques, no one definition had been developed previous to Volker's work. He developed the notion of a fuzzy random variable from the perspective of set theory, using the general topology approach and results from the theories of topological measure and analytic spaces. In fixed point, random fuzzy spaces do not introduce any outcomes.

1.12 A FIXED POINT THEOREM IN CONE METRIC SPACE

A famous issue in metric spaces is the investigation of fixed points for contractive mappings, and Huang and Zhang's introduction of the cone metric space is one such generalization. They gave some essential results for a self-map meeting a contractive condition in this space and replaced the set of real numbers from a metric space with an ordered Banach space.

This is a significant step in the development of cone metric space fixed point theory. Ali Abou Bakr, S,M, given their contribution in cone metric space fixed point theory[4].Verma, Kabir, Chauhan and shrivastava has generalized fixed point theorem for multi-valued contractive mapping in cone b-metric space[50]

1.13 COUPLED FIXED POINT IN TWO G- METRIC SPACE

Generalised metric space was first introduced in 2006 by Mustafa and Sims, who also provided several fixed point theorems in G-metric space.

V. Lakshmikantham developed the idea of a linked coincidence point of mapping. They also looked at a few fixed point theorems in partially ordered metric spaces. In generalised metric spaces, linked coincidence fixed point theorems in

2011.

The investigation of common fixed point theory in G-metric spaces was started by Abbas and Rhoades[24] in 2009.Recent common fixed point theorems were provided in two G-metric spaces in a fundamentally new and more organic method by Feng gu. In 2016, Rahim Shah, Akbar Zada, and Tongxing Li[28] presented the idea of integral type contraction regarding generalized metric space and demonstrated some novel common coupled coincidental fixed point results of integral type contractive mappings in generalized metric space. Latif, Nazir and Abbas presented the stability of fixed points in generalized metric spaces[27]. Pathak & Gharib, Malkawi, Rabaiah, Shatanawi and Alsauodi have given their contribution to a fixed point theorem in matric space[17,29]. Hong, Pasman, Quddus and Mannan has contribute to supporting risk management decision making by converting linguistic graded qualitative risk matrices through interval type-2 fuzzy sets[22]. Fixed point theorems for nonlinear contractive mappings inordered b-metric space with auxiliary function[38]. Ansari, Chandok, Hussain, Mustafa and Jaradat has proved some common fixed point theorems for weakly α-admissible pairs in G-metricspaceswith auxiliary function[5].

Banach space, Hilbert space, fuzzy set, fuzzy subset generated by mapping, fuzzy real numbers, fuzzy metric spaces, fuzzy normed linear spaces, fuzzy uniform spaces, fuzzy metric spaces with respect to continuous t-norm, fuzzy 2-metric spaces, fuzzy 3-metric spaces, two generalized metric spaces, random fuzzy metric spaces, cone metric spaces, and generalized metric spaces are some of the concepts that have been defined and the results that follow them.

1.14 BANACH SPACE

Over the past three decades, there has been a lot of research done on fixed point theorems and common fixed point theorems satisfying contractive type conditions. Polish mathematician Banach established a theorem in 1922 that guarantees the existence and uniqueness of a fixed point under approximation conditions. The Banach fixed point theorem or Banach contraction principle are two names for his conclusion. This theorem offers a method for resolving several applicable issues in engineering and the mathematical sciences. Numerous researchers have improved, expanded, and generalized.

Banach's fixed point theorem in different ways.

Definition 1.14.1: Let X be a linear space (= vector space) over the flied f of complex scalars. Then X is a normed linear space if for every $f \in X$ there is a real number $||f||$ called the norm of f such that:

a. $||f|| \geq 0,$

b. $||f|| = 0$ if *and only* if $f = 0,$

c. $||c\,f|| = ||c||\ ||f||$ for every scalar c,

d. $||f + g|| \leq ||f|| + ||g||.$

Definition 1.14.2: Let X be a normed space and let $\{f_n\}_{\in \rho}$ be a sequence of elements of X.

a. $\{f_n\}_{n\in\rho}$ Converges to $f \in X$ if $\lim_{n\to\infty} ||f - f_n|| = 0, \exists\, N > 0,\ \forall\ \varepsilon > 0, n \geq N, ||f - f_n|| < s$. In this case we write $lim_{n\to\infty} f_n = f\ or\ f_n \to f.$

b. $\{f_n\}_{n\in\rho}$ is Cauchy if $\forall\ \varepsilon > 0, \exists\, N > 0, \forall\, m, n \geq N, ||f - f_n|| < \varepsilon.$

Definition 1.14.3: Any convergent sequence in a normed linear space is easily demonstrated to be a Cauchy sequence. The statement that all Cauchy sequences converge in any normed linear space may or may not be accurate. A normed linear space X is considered complete if and only if it possesses the property that all Cauchy sequences converge. A Banach space is a fully normed linear space.

1.15 HILBERT SPACE

German mathematician David Hilbert (1862–1943), who made several contributions to the growth of mathematics, is best remembered for his groundbreaking work in the area of functional analysis. The mathematical formulation of quantum theory relies heavily on the notion of Hilbert space. David Hilbert, who developed the idea in the setting of integral equations, is the name given to these spaces.

Definition 1.15.1: Let H be a vector space. Then H is an inner product space if for every $f, g, \in X$ there exists a complex number (f, g) called the inner product of f and g such that:

(a) (f, f) is real and $(f, f) \geq 0.$

(b) $(f, f) = 0$ if and only if $f = 0$.

(c) $(g, f) = \overline{< f, g >}$

(d) $(af_1 + bf_2, g) = a(f_1, g) + b\,(f_2, g)$

Each inner product determines a norm by the formula $||f|| = (f, f)^{½}$. Hence every inner product space is a normed linear space. The Cauchy- Schwarz inequality states that $| < f, g > | \leq ||f||\ \ ||g||,\ \forall\ f, g\ \in H$.

A Hilbert space is one in which an inner product space H is complete. A Hilbert space is, in other words, a Banach space whose norm is defined by an inner product.

1.16 METRIC SPACE

Definition 1.16.1: Let X be any set. Let $d(x, y)$ be a function defined on the set X $\times$ X satisfying the following condition:

1. $d(x,y) \geq 0$
2. $d(x,\ y) = 0$ if and only if $x = y$
3. $d(x,y) \leq d(x,z) + d(z,y)$ *triangle inequality.*

Such a function $d(x, y)$ is called a metric space on X, it is a mapping of X $\times$ $X \rightarrow R$. A set X with a metric d is called a metric space.

Definition 1.16.2: A Sequence of points $\{X_n\}$ is said to converge to a point x of X if $d\ (X_n, X) \rightarrow 0$ as n $\rightarrow \infty$ there exists an integer n_o depends on ϵ, such that $0 \leq d(X_n, X) < \epsilon$, for each $n \geq n_0$. The point x is called the limit of the sequence.

Definition 1.16.3: A sequence $\{X_n\}$ in a metric space (X, d) is called a Cauchy sequence if $\forall\ \varepsilon > 0$ there exists an integer n_o such that $d(x_n, x_m) < \varepsilon$ for each $m > n \geq n_0$.

Definition 1.16.4: Let f and g be self maps of a metric space (X, d) then f and g are said to commuting if and only if $fg = gf$.

Definition 1.16.5: Let f and g be self maps of a metric space (X, d). A point $x \in X$ is said to be a coincidence point of f and g if $fx = gx$

Definition 1.16.6: Two self-maps S and T of a metric space (X, d) are said to be compatible if $lim_{n\rightarrow\infty}\ d(STx_n, TSx_n) = 0$ whenever $\{X_n\}$ is a sequence in X such that $lim_{n\rightarrow\infty}\ Sx_n = lim_{n\rightarrow\infty}\ Tx_n = t$, for $t\ \in$ X.

1.17 FUZZY SET

Lotif. A. Zadeh was the first to propose the origins of the fuzzy set. The outcomes of fuzzy sets used in our research are presented here without justification. We consult the relevant sources for more information.

Definition 1.17.1: A function A from a non-empty set X into unit interval I is called a fuzzy set in X. $\forall x \in A, A(x)$ is called the grade of membership of x in A. A is also said to be a fuzzy subset of x.

Definition 1.17.2: A fuzzy subset A is said to be empty if $A(x) = 0$ for all x in X and A is whole if A(x) =1 for all x in X. The empty fuzzy set is denoted by Ø or 0 and whole set is denoted by X or 1.

Definition 1.17.3: Let X be a set and A and B be two fuzzy subsets of X. A is said to be included in B if $A(x) \leq B(x)$for all $x \in$ X. It is denoted by $A \leq B$.

(a) A is said to be equal to B if A(x) = B (x) for all $x \in$ X and written as A =B.

(b) B is said to be complement of A if B (x) =1 - A (x) for every $x \in$ X and denoted by $B = A^c. Obviously\ (A^c)^C = A$.

Definition 1.17.4: The union of A and B be defined by $\mathrm{A \cup B}\,(x) = \max\{\mathrm{A}(x), \mathrm{B}(x)\}$ or $\mathrm{A}(x) \cup \mathrm{B}(x)\ \forall\ x \in \mathrm{X}$.

The intersection of A and B be defined by$A \cap B\ (x) = \min\{A(x), B(x)\}\ or\ A(x) \cup B(x)\ \forall\ x \in X.$

The difference of A and B is defined by $A \backslash B = A \cap B$.

In general, if I is an index set and $A = \{A_\alpha \backslash \alpha \in I\}$ is a family of fuzzy subsets of X, then their union UA_α is defined by $(UA_\alpha)(x) = \sup\ \{A_\alpha(x) \backslash \alpha \in I\}, x \in X$ and the intersection $\cap\ A_\alpha$ is defined by $(\cap\, A_\alpha\ (x) = \inf\ \{A_\alpha(x) \backslash \alpha \in I\}, x \in X$.

1.18 FUZZY REAL NUMBER

Definition 1.18.1: A fuzzy real number x is a fuzzy set on the real axis i.e. a mapping $x: R \rightarrow [0,1]$ associating with each real number t its grade of membership x(t).

Definition 1.18.2: A fuzzy real number x is convex if $x(t) \geq x(s) \wedge x(r) = \min\{x(s), x(r)\}$ where $s \leq t \leq r$.

Definition 1.18.3: The α -level set of a fuzzy real number x is denoted by $[x]\alpha = \{t: x(t) \geq t\}, 0 < \alpha \leq 1$.

A fuzzy real number x is called normal if $x(t_0) = 1$ for some $t_0 \in R$. x is called upper semi continuous if $\forall$ S > 0, $x^{-1}([0, \alpha + s]), \forall \alpha \in I$ is open in the usual topology of R. It can be easily seen that the $a - level\ set\ [x]_\alpha$ of an upper semi continuous, normal, convex fuzzy real number x for each $\alpha, 0 < \alpha \leq 1$ is closed interval $[\alpha^\alpha, b^\alpha]$ where $\alpha^\alpha = -\infty\ and\ b^\alpha = \infty$ are also admissible. The set of all upper semi continuous, normal, convex fuzzy real number is denoted by E or R (I). Since each $r \in R$ can be considered as a fuzzy real number $\bar{r}$, denoted by $r(t) = 1, \{t = 0, t \neq r\ then\ \bar{r} \in E\}$. in other words R can be embedded in E or R(I).

Definition 1.18.4: A fuzzy real number x is called non negative if $x(t) = 0$ for all $t < 0$.

The set of all non negative fuzzy real number is denoted by G or R*(I).

The equality of fuzzy real number x and y is defined by $x(t) = y(t)$ for all $t \in R$.

Definition 1.18.5: For $k \in E, k_x(t) = x\ (k^{-1}t)$ and $0x$ is defined to be $\bar{0}$.

Definition 1.18.6: A partial ordering $' \leq'\ in\ E$ is defined by $x \leq y$ if and only if

$a_1^\alpha \leq a_2^\alpha$ and $b_1^\alpha \leq b_2^\alpha$ for all $\alpha \in (0,1)$, where $[x] = [a_1^\alpha, b_1^\alpha]$ and $[y]\alpha = [a_2^\alpha, b_2^\alpha]$.

Definition 1.18.7: A sequence $\{x_n\}$ in E converges to X $\in$ E, denoted by $lim_{n\to\infty}$ $X_n = x$ if $lim_{n\to\infty}\ a_n^\alpha = lim_{n\to\infty}\ b_n^\alpha = b^\alpha\ \forall\ \alpha \in (0,1)$, where $[X_n]\ \alpha = [a_n^\alpha, b_n^\alpha]$ and $[X]\ \alpha = [a^\alpha, b^\alpha]$.

1.19 FUZZY METRIC SPACE

Definition 1.19.1: Let X be a non empty set, d a mapping from X×X into G or R*(I) and let the mappings $L, R: [0,1] \times [0,1] \to [0,1]$ be symmetric, non decreasing in both arguments and satisfy $L\ (0,0) = 0$ and $R\ (1,1) = 1$ Denoted $[d(x,y)]\ \alpha = [\lambda_\alpha\ (x,y),\ p_\alpha\ (x,y)]$ for $x,\ y \in X, 0 < \alpha \leq 1$.

Remark: λ_α(x, y) is called left end point and ρ_α(x, y) is right end point of α-level set of d(x,y).

The quadruple (X, d, L, R) is called a fuzzy metric space and d, a fuzzy metric if

i. $d(x,\ y) = 0$ if and only if $x = y$.

ii. $d(x,y) = d(y,x)$ for all $x, y, \in$ X.

iii. $\forall\, x,\ y, z \in$ X

a. $d(x,y)(s+t) \geq L\big(d(x,z)(s),\ d(z,y)(t)\big)$ whenever $s \leq \lambda_1(x,z), t \leq \lambda_1\,(z,y)$ and $s+t \leq \lambda_1(x,y)$

b. $d(x,y)(s+t) \geq R\big(d(x,z)(s), d(z,y)(t)\big)$ whenever $s \leq \lambda_1(x,z), t \leq \lambda_1(z,y)$ and $s+t\ \leq\ \lambda_1(x,y)$

The usual metric space is special case of the fuzzy metric space.

Lemma 1.19.1: The triangle inequality (iii) (b) with R=Max is equivalent to the triangle inequality $p_\alpha(x,y) \leq\ p_\alpha(x,z) +\ p_\alpha(z,y)$for all α ∈ $[0,1]$ and x , y, z ϵ X.

Lemma 1.19.2: The triangle inequality (iii) (a) with L=Min is equivalent to the triangle inequality $\lambda_\alpha(x,y)\ \leq\ \lambda_\alpha\,(x,z) + \lambda_\alpha\,(z,y)$for all α ϵ $(0,1)$ and x, y, z ϵ X.

Theorem 1.19.1: In a fuzzy metric space (X, d, Min, Max) the triangle inequality is equivalent $d(x,y) \leq d(x,z) + d(z,y)$ for x, y, z ϵ X.

Definition 1.19.2: Let (X, d, L, R) be a fuzzy metric space. A sequence $\{x_n\}$ is said to be convergent to $x \in$ X if and only if $lim_{n\to\infty}(x_n,x) =\ \bar{0}$ and denoted by $lim_{n\to\infty}\,x_n =\ x$.

Lemma 1.19.3: A sequence $\{x_n\}$ in a fuzzy metric space (X, d, L, R) converges to $x \in$ X if and only if $lim_{n\to\infty}(x_n,x) = 0$, α ∈ $(0,1)$.

Definition 1.19.3: A sequence $\{x_n\}$ in a fuzzy metric space (X, d, L, R) is called a Cauchy sequence if $lim_{m,n\to\infty}d\,(x_m,x_n) =\ \bar{0}$.

Lemma 1.19.4: A sequence $\{x_n\}$ in a fuzzy metric space (X, d, L, R) is called a Cauchy sequence if $lim_{m,n\to\infty}\,p_\alpha\,(x_m,x_n) = 0$.

Definition 1.19.4: If every Cauchy sequence in a fuzzy metric space X converges to a point in X, then X is said to be complete.

1.20 FUZZY NORMED LINEAR SPACE

Definition 1.20.1: Let X be a vector space over R. Let $||.||$: X $\to E$ and let the mapping L, R: $[0,1] \times [0,1] \to [0,1]$ be symmetric, non decreasing in both argument and satisfy L(0, 0) = 0 and R(1,1)=1.

We write $[||x||]\alpha = [|| \, |x| \, ||^{\alpha}, \; || \, |x| \, ||^{\alpha}]$ for $x \in X, 0 < \alpha \leq 1$ and suppose for all $x \in X, x \neq 0$

there exists $a_0 \in X, [0,1]$ independent of x such that for all $a \leq a_0$

a. $|| \, |x|^2 \, ||^{a} < \infty$

b. $\inf || \, |x|^1 \, ||^{a} > 0$

The quadruple $(X, ||.||, L, R)$ fuzzy norm, if

(i) $||x|| = \bar{0}$ if and only if $x = 0$.

(ii) $||rx|| = |r| \; ||x||, x \in X, r \in R$.

(iii) $\forall x, y \in X$,

a. Whenever $s \leq \forall || \, |x| \, ||_1^1, t \leq || \, |y| \, ||_1^1$ and $s + t \leq || \, |x + y| \, ||_1^1 \; ||x + y|| \; (s + t) \geq L \, (||x|| \, (s), ||x|| \, (t))$

b. Whenever $s \geq || \, |x| \, ||_1^1, t \geq || \, |y| \, ||_1^1$ and $s + t \geq || \, |x + y| \, ||_1^1 \; ||x + y|| \, (s + t) \leq R \, (||x|| \, (s), ||x|| \, (t))$

Definition 1.20.2: Let $(X, ||.||)$ is a fuzzy normed linear space. A sequence $\{X_n\} \in X$ is said to converge to $x \in X$, denoted by $lim_{n\to\infty} x_n = x$ if and only if $lim_{n\to\infty} \left||x_n - x\right|| = \bar{0}$, i.e. $lim_{n\to\infty} || \, |x_n - x| \, ||_1^{\alpha} = lim_{n\to\infty} || \, |x_n| \, ||_2^{\alpha} = 0 \; \forall \alpha \in (0,1)$.

Definition 1.20.3: A sequence $\{X_n\}$ in a fuzzy normed linear space $(X, ||.||)$ is called Cauchy sequence if $lim_{n\to\infty} \left||x_m - x_n\right|| = \bar{0}. i.e.$ if $lim_{m,n\to\infty} || \, |x_m - x_n| \, ||^{\alpha} = \bar{0}, \forall \alpha \in (0,1)$.

1.21 FUZZY UNIFORM SPACE

In function analysis, uniform spaces lie between metric space and general topological space.The same concept applies in fuzzy structure.

Definition 1.21.1: A uniform space X is sequentially complete if each Cauchy sequence in X converges in X.

Definition 1.21.2: Let (X, d, L, R) be a fuzzy metric space with $lim_{a\to0^+} R(a, a) = 0$. If (X, d, L, R) is a complete fuzzy metric space then (X, V) is a sequentially complete uniform space.

Result 1.21.3: Family of fuzzy metrics induced by fuzzy metric space (X, d, L, R): In a fuzzy metric space (X, d, L, R) the family $\{p_\alpha : \alpha \in (0,1)\}$ satisfies the following condition:

i. for all $\alpha \in (0,1)$and $x \in X,\ p_\alpha(x,x) = 0;$

ii. for all $\alpha \in (0,1\}$ and $x, y \in X, p_\alpha\ (x,y) = p_\alpha\ (y,x);$

iii. if $p_\alpha\ (x,y) = 0$ for all $\alpha \in (0,1),$ then $x = y$.

If R=Max, then p_α also satisfies $p_\alpha(a,y) \le p_\alpha(x,z) + p_\alpha(z,y)$

If $lim_{t\to\infty}\ d(x,y)\ (t) = 0, \forall\ x,\ y \in X,$ then $p_\alpha(x,y) < \infty, \forall\ \alpha \in (0,1)$ and x, y .

Hence if (X, d, L, R) is a fuzzy metric space with $lim_{t\to\infty}\ d\ (x,y)(t) = 0\ \forall x, y \in X$ then the family p_α (x, y) $= 0, \forall\ \alpha \in (0,1),$ then $x = y$.

Theorem 1.21.4: if (X, d, L, Max) is a fuzzy metric space then for any $\alpha \in (0,1)$

i. $u(\varepsilon_1, \alpha) \le u\ (\varepsilon_2, \alpha) for\ 0 \le \varepsilon_1 \le \varepsilon_2\ and$

ii. $u(\varepsilon_1, \alpha)\ .u\ (\varepsilon_2, \alpha) \le u\ (\varepsilon_1 + \varepsilon_2\ \alpha)\ for\ any\ E_1 E_2 > 0$

where $\mathrm{u}\ (\varepsilon_1, \alpha). \mathrm{u}(\varepsilon_2, \alpha) = \{(\mathrm{x}, \mathrm{y}) : \varepsilon z \in \mathrm{X}$ with $(\mathrm{x}, \mathrm{z}) \in \mathrm{u}\ (\varepsilon_1, \alpha)\ and\ (\mathrm{z}, \mathrm{y}) \in \mathrm{u}\ (\varepsilon_2, \alpha)\}$

1.22 FUZZY METRIC SPACE WITH RESPECT TO CONTINUOUS T-NORM

Definition 1.22.1: A binary operation *: [0, 1] × [0, 1] → [0, 1] is said to be continuous t-normif it satisfies the following condition:

i. * is commutative and associative;

ii. * is continuous;

iii. $a * 1 = a$ for $a \in [0,1]$.

iv. $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Examples of continuous t-norms are Lukasiewicz t-norm, that is, $a_L^+ b = \max\{a + b - 1, 0\}$, product t-norm, that is, $a_P^*\ b = ab, and\ minimum\ t - norm, that\ is, a_M^* b = \min\{a, b\}$.

Example 1.22.1: Define $a * b = \min\ (a,b)\ ,\forall\ a,\ b\ \in [0,1].\ then * is\ a\ continuous\ t-norm.$

Remark 1.22.1 Given an arbitrary set X, a fuzzy set M on X is function from X to the unitinterval $[0,1]. Let\ [0,1]^X = \{f : X \to [0,1]\}$ thus $M\ \in [0,1]^X$.

Definition 1.22.2: A fuzzy metric space is a triple (X; M; *), where X is a non-empty set, * is acontinuous t-norm and M is a fuzzy set on $X \times X \times [0,+\infty]$, satisfying the following properties:

1. $M\ (x,\ y,\ 0) = 0$ for all $x; y\ \in X$;
2. $M\ (x,\ y,\ t) = 1$ for all $t > 0$ if f $x = y$;
3. $M\ (x,\ y,\ t) = 1$ for all $t > 0$ if f $x = y$;
4. $M\ (x,\ y,\ \cdot): [0,+\infty] \to [0,1]$ is left continuous for all $x, y\ \in X$;
5. $M\ (x,\ z,\ t+s)\ \geq M(x,\ y,\ t) * M(y,z,s)$ for all $x,\ y,\ z\ \in X$ and for all $t,\ s > 0$;

Remark 1.22.2: Definition 1.16.4 (4) means that for each x, y ∈ X there is a function $M_{xy}: [0,\infty] \to I, t\ \to M(x,y,t), M\ (x,y,t)$ Can be notion of as the

degree of nearness between a and y with respect to $t \geq 0$.

We can fuzzify example of metric spaces into fuzzy metric space in a natural way:

Example 1.22.2: Let (X, d) be a metric space define $a * b = \min\ \{a,b\}\ \forall\ a,\ b\ \in X$, we have $M(x,y,t) = \frac{t}{t=d(x,y)}$ for all x, y in X and t > 0. Then (X, M, *) is fuzzy metric induced by a metric d is called the standard fuzzy metric space.

The definition that follows is an adjustment to Definition 1.16.4. This adjustment is required since the fuzzy metric in Definition 1.16.4 does not produce a Hausdorff topology.

Definition 1.22.3: The 3-tuple (X, M,*) is called a fuzzy metric space (shortly FM-space) if X is an arbitrary set * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty]$ satisfying the following conditions: for all x, y, z in X and s, t >0,

i. M(x, y, t) > 0;
ii. M(x, y, t)=1, for all t > 0 if f x=y,
iii. M(x, y, t)=M(y, x, t),

iv. M(x, y, t)*M(y, z, s) ≤ M(x, z, t+s)

v. $M(x, y): [0, \infty) \rightarrow [0,1]$ is left continuous.

vi. $\lim_{t\rightarrow\infty} M(x, y, t) = 1$ For all x, y in X.

In the sequel the fuzzy set M as in Definition 1.16.7 will be referred to as a fuzzy metric.It shall be shown that the topology induced by the fuzzy metric space (X, M, *) is Hausdorff.

Lemma 1.22.1: M(x, y,.) is non decreasing for all x, y in X.

Definition 1.22.4: A sequence $\{X_n\}$ is a fuzzy metric space (X,M,*) is said to converge to $x \in X$ if for each $s, 0 < s < 1$ and $t > 0, n_o \in N$ such that

$M(x_n, x, t) > 1 - s, \forall n \geq n_0$.

Definition 1.22.5: A sequence $\{X_n\}$ is a fuzzy metric space (X, M, *) is said to be a Cauchy sequence if for each $s, 0 < s < 1$ *and* $t > 0$, $n_0 \in N$ *such that* M (x_n, x_m, t) >1-ε, ∀ n, m ≥ n_0.

Definition 1.22.6: A fuzzy metric space is said to be complete if every Cauchy sequence isconvergent.

Definition 1.22.7: A function M is continuous in fuzzy metric space if $x_n X, y_n \rightarrow y$ then $\lim_{n\rightarrow\infty} M(x_n, y_n, t) = M(x, y, t)$ for $t > 0$.

1.23 TOPOLOGY AND FUZZY METRIC SPACES

We continue to present some concept and results from classical metric spaces theory in thecontext of fuzzy metric space.

Definition 1.23.1: Let (X, M, *) be a fuzzy metric space. We define the open ball B (x, r, t) with centre $x \in X$ and radius $r, 0 < r < 1, t > 0$, as B $(x, r, t) = \{y \in X: M(x, y, t) > 1 - r\}$

Definition 1.23.2: A subset A of a fuzzy metric space (X, M, *) is said to be open if given any point $a \in A$, there exists $0 < r < 1, t > 0$ such that $B(a, r, t) \leq A$.

Theorem 1.23.1: Every open ball in a fuzzy metric space (X, M, *) is an open set.

Theorem 1.23.2: Every fuzzy metric space is Hausdorff.

Preposition 1.23.1: Let (X, d) be a metric space and $M_d(x, y, t) = \frac{t}{t=d(x,y)}$

Corresponding standard fuzzy metric space on X. Then the topology T_d induced by the metric d and thetopology T_{Md} induced by the fuzzy metric space M_d are the same that is $T_d = T_{Md}$.

Definition 1.23.3: Let (X, M, *) be a fuzzy metric space. A subset A of X is said to be F-bounded if there exists t > 0 and 0 < r < 1 such that $M(x, y, t) > 1 - r$ for all $x, y, \in A$.

Remark 1.23.1: Let (X, M, *) be a fuzzy metric space induced by a metric d on X. then $A \leq X$ is F-bounded if and only if it is bounded. This is what R lower would call a goodextension of the notion of bounded.

Theorem 1.23.3: Every compact subset A of a fuzzy metric space X is F- bounded.

Remark 1.23.2: In a fuzzy metric space every compact subset is closed and bounded.

Theorem 1.23.4: Let (X, M, *) be a fuzzy metric space and T_M be the topology induced bythe fuzzy metric. Then for a sequence $\{X_n\}$ in X, the sequence $\{X_n\}$ converges to x if and only if (X_n, X, t) converges to 1 as n tends to ∞.

Remark 1.23.3: Let (X, M, *) be a fuzzy metric space and $\{X_n\}$ be a sequence in X. Then $lim_{n\to\infty} d(x, y) = 0$ if and only if $lim_{n\to\infty} M_d (x_n, x, t) = 1 \; for \; all \; t > 0 \; and \; x \in X.$

Definition 1.23.4: Let (X, M, *) be a fuzzy metric space. We define a closedball $B[x, r, t]$ with center $x \in X$ and radius $r, 0 < r < 1, \; t > 0 \; as \; B[x, r, t] = \{y \in X: M(x, y, t) \geq 1 - r\}$

Lemma 1.23.1: Every closed ball in a fuzzy metric space (X, M, *) is closed set.

Theorem 1.23.5: Let (X, M, *) be a complete fuzzy metric space. Then the intersection of acountable number of dense open set is dense.

1.24 FUZZY 2-METRIC SPACE

Here we recall the definition of fuzzy 2-metric space.

Definition 1.24.1: An operation $*: [0,1]^3 \to [0,1]$ is called a continuous t-norm if $([0,1] \, *)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$ for all $a_1, a_2, b_1, b_2 \; and \; c_1, c_2$ are in [0,1].

Definition 1.24.2: The triple (X, M, *) is called a fuzzy 2-metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^3 \times [0,1]$ satisfying the following condition:

i. $M(x, y, z, 0) = 0$

ii. $M(x, y, z, t) = 1; t > 0$ and when at least two of the three points are equal.

iii. M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t) (symmetry about three variables)

iv. $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) + M(x, u, z, t_2) + M(u, y, z, t_3)$ (This corresponds to tetrahedron inequality in 2-metric space).

v. $M(x, y, z, .): [0, \infty] \rightarrow [0,1]$ *is left continuous*; $\forall\, x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$.

Note 1.24.1: The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle formed by the three points x, y, z < t.

Definition 1.24.3: A function M is continuous in fuzzy 2-metric space if $x_n \rightarrow x, y_n \rightarrow y$ then $lim_{n\rightarrow\infty}(x_n, y_n, a, t) = M(x, y, a, t)$ for all $a \in X$ *and* $t > 0$.

Lemma 1.24.1: Let (X, M, *) be a fuzzy 2-metric space. Then M(x, y, z, ·) is non-decreasing function for all $x, y, z \in X$.

Definition 1.24.4: A sequence $\{x_n\}$ in a fuzzy 2-metric space (X, M, *) converge to a point x in X if and only if $lim_{n\rightarrow\infty} M(x_n, x, a, t) = 1,$ for all $a \in X$ *and* $t > 0$.

Definition 1.24.5: let (X, M, *) be a fuzzy 2-metric space. A sequence $\{x_n\}$ is called a Cauchy sequence, if and only if $lim_{n\rightarrow\infty} M\left(x_{n+p}, x_n, a, t\right) = 1,$ for all $a \in X$ *and* $t > 0, p > 0$.

Definition 1.24.6: A fuzzy 2-metric space (X, M, *) is said to be complete if and only if every Cauchy sequence in X converges in X.

Note 1.24.2: Let (X, M, *) be a fuzzy 2-metric space, then $lim_{n\rightarrow\infty} M(x, y, z, t) = 1$.

Lemma 1.24.2: If for all $x, y, z \in X, t > 0$ and $0 < k < 1, M(x, y, z, kt) \geq M(x, y, z, t)$ then $x = y$.

1.25 FUZZY 3-METRIC SPACE

Definition 1.25.1: A operation $*: [0,1]^4 \rightarrow [0,1]$ is called a continuous t-norm if $([0,1],*)$ is abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 * d_1 \leq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2 \forall a_1, a_2, b_1, b_2, c_1, c_2,$ $and\ d_1, d_2\ are\ in\ [0,1]$.

Definition 1.25.2: The 3-tuple (X, M, *) is called a fuzzy 3-metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^4 \times [0,1]$ satisfying the following condition:

i. $\mathrm{M}\ (\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{w}, 0) = 0$

ii. $\mathrm{M}\ (\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{w}, \mathrm{t}) = 1; \mathrm{t} > 0$

iii. $\mathrm{M}\ (\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{w}, \mathrm{t}) = \mathrm{M}\ (\mathrm{x}, \mathrm{w}, \mathrm{z}, \mathrm{y}, \mathrm{t}) = \mathrm{M}\ (\mathrm{y}, \mathrm{z}, \mathrm{w}, \mathrm{x}, \mathrm{t}) = \mathrm{M}\ (\mathrm{z}, \mathrm{w}, \mathrm{x}, \mathrm{y}, \mathrm{t}) = \cdots$

iv. $M(x, y, z, w, t_1 + t_2 + t_3 + t_4) \geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2) * M(x, u, z, w, t_3) * M(u, y, z, w, t_4)$

v. $M(x,\ y,\ z,\ w,\ .): [0, \infty) \rightarrow [0,1]\ is\ left\ continuous; for\ all$

$x, y, z, u, w\ \in X\ and\ t_1, t_2, t_3, t_4 > 0.$

Definition 1.25.3: A sequence $\{x_n\}$ in a fuzzy 3-metric space (X, M, *) converge to a point x in X if and only if $lim_{n\rightarrow\infty} M(x_n, x, a, b, t) = 1$, for all $a, b, \in X$ and $t > 0, p > 0$.

Definition 1.25.4: Let (X, M, *) be a fuzzy 3-metric space. A sequence $\{x_n\}$ is called a Cauchy sequence if f $lim_{n\rightarrow\infty} M\left(x_{n+p}, x_n, a,\ b,\ t\right) = 1$, for all $a, b, \in X$ and $t > 0, p > 0$.

Definition 1.25.5: A fuzzy 3-metric space (X, M, *) is said to be complete and only if every Cauchy sequence in X converges in X.

Definition 1.25.6: A function M is continuous in fuzzy 3-metric space if $x_n \rightarrow x, y_n \rightarrow y$ then $lim_{n\rightarrow\infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t)$ for all $a, b, \in X$ and $t > 0$.

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